The Complete Closure of a Graph

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ABSTRACT

We define the complete closure number \( cc(G) \) of a graph \( G \) of order \( n \) as the greatest integer \( k \leq 2n - 3 \) such that the \( k \)th Bondy–Chvátal closure \( Cl_k(G) \) is complete, and give some necessary or sufficient conditions for a graph to have \( cc(G) = k \). Similarly, the complete stability \( cs(P) \) of a property \( P \) defined on all the graphs of order \( n \) is the smallest integer \( k \) such that if \( Cl_k(G) \) is complete then \( G \) satisfies \( P \). For some properties \( P \), we compare \( cs(P) \) with the classical stability \( s(P) \) of \( P \) and show that \( cs(P) \) may be far smaller than \( s(P) \). © 1993 John Wiley & Sons, Inc.

1. INTRODUCTION AND NOTATION

All the graphs \( G = (V,E) \) we consider are undirected and simple, of order \( |V| = n \) and size \( m \). The degree in \( G \) of a vertex \( u \) is denoted by \( d_G(u) \) or by \( d(u) \) when there is no ambiguity. The complete graph of order \( n \) is denoted by \( K_n \), an independent set of \( n \) vertices by \( K_n \), a simple cycle of length \( l \) by \( C_l \), and a simple path on \( l \) vertices by \( P_l \). The notation \( G \cup H \) means the disjoint union of the two graphs \( G \) and \( H \), and \( G + H \) the disjoint union of \( G \) and \( H \) plus all the edges between \( G \) and \( H \). When we say that \( G \) contains a graph \( H \), we mean that \( H \) is a subgraph of \( G \).

A graph \( G \) of order \( n \) is hamiltonian if it contains a cycle \( C_n \), pancyclic if it contains cycles \( C_l \) of every length between 3 and \( n \), and hamilton-connected if for any pair of vertices \( x \) and \( y \) there exists a hamiltonian path with endvertices \( x \) and \( y \). It is s-hamiltonian (respectively \( s \)-hamiltonian-connected) if the deletion of any \( s \) vertices produces a hamiltonian graph.
(respectively a hamilton-connected graph). It is s-edge-hamiltonian if for any set \( F \) of \( s \) edges that form pairwise vertex disjoint paths in \( G \), \( G \) has a hamiltonian cycle containing all the edges of \( F \).

We denote by \( \alpha(G) \) the maximum cardinality of an independent set of vertices and by \( \mu(G) \) the smallest number of pairwise disjoint paths covering all the vertices of \( G \).

Other notation and terminology can be found in [4].

In [3], Bondy and Chvátal introduced the closure of a graph and the stability of a property. The \( k \)-closure \( \text{Cl}_k(G) \) is obtained from \( G \) by recursively joining pairs of nonadjacent vertices whose degree sum is at least \( k \), until no such pair remains. The \( k \)-closure is independent of the order of adjacency of the edges. It is clear that any graph \( G \) of order \( n \) satisfies

\[
G = \text{Cl}_{2n-3}(G) \subseteq \text{Cl}_{2n-4}(G) \subseteq \cdots \subseteq \text{Cl}_1(G) \subseteq \text{Cl}_0(G) = K_n.
\]

A property \( P \) defined on all the graphs of order \( n \) is said to be \( k \)-stable if for any graph of order \( n \) that does not satisfy \( P \), the fact that \( uv \) is not an edge of \( G \) and that \( G + uv \) satisfies \( P \) implies \( d_G(u) + d_G(v) < k \). Every property is \((2n-3)\)-stable and every \( k \)-stable property is \((k+1)\)-stable. We denote by \( s(P) \) the smallest integer \( k \) such that \( P \) is \( k \)-stable and call it the stability of \( P \). This number usually depends on \( n \) and is at most \( 2n - 3 \).

The closure theory is based on the fact that if the property \( P \) is \( k \)-stable and if \( \text{Cl}_k(G) \) satisfies \( P \), then \( G \) itself satisfies \( P \). But it is not always easier to check a property \( P \) in \( \text{Cl}_k(G) \) than in \( G \), and since the complete graph \( K_n \) has a lot of interesting properties, this theory is often used in a weaker form, by proving that \( \text{Cl}_k(G) \) is complete. This remark leads us to introduce the complete closure number of a graph and the complete stability of a property of graphs.

The complete closure number \( cc(G) \) of a graph \( G \) of order \( n \) is the greatest integer \( k \leq 2n - 3 \) such that \( \text{Cl}_k(G) = K_n \). For example, \( cc(K_n) = 2n - 3 \), \( cc(K_n - e) = 2n - 4 \), \( cc(K_n^+) = 0 \), and \( cc(G) = 2d \) if \( G \) is \( d \)-regular.

The complete stability \( cs(P) \) of a property \( P \) defined on all graphs of order \( n \) and satisfied by \( K_n \) is the smallest integer \( k \) such that any graph \( G \) satisfies \( P \) if \( \text{Cl}_k(G) \) is complete. This number usually depends on \( n \) and satisfies \( cs(P) \leq s(P) \).

The purpose of this paper is to study these notions and to compare \( cs(P) \) with \( s(P) \) for some properties \( P \). For instance, it is well known that the property "\( G \) is hamiltonian" has stability \( n \). But although many conditions that are sufficient for a graph to be hamiltonian are also sufficient for the graph to be pancyclic, the stability of the property "\( G \) is pancyclic" is unknown (however, it is easy to prove that the property "\( G \) contains a triangle" has stability \( 2n - 3 \)). That "\( G \) is pancyclic" has complete stability at most \( n + 1 \) (see Theorem 4.1) justifies our interest in \( cs(P) \).

It would be interesting to know which cycles are necessarily contained in a graph \( G \) of order \( n \) whose \( n \)-closure is complete. This condition is
Weaker than Ore’s condition, \( d_G(u) + d_G(\nu) \geq n \) for every nonedge \( uv \) of \( G \), which implies that \( G \) is pancyclic or bipartite [2]. A partial answer to this question, obtained in [7], will be recalled in Theorem 4.4.

In Section 2 we characterize the graphs \( G \) satisfying the Ore condition \( d_G(u) + d_G(\nu) \geq k \) for each nonedge \( uv \) of \( G \) and \( \text{Cl}_{k+1}(G) \neq K_n \). In Section 3 we give some necessary conditions for a graph to have its \( k \)-closure complete. This leads to statements of the form “If \( \text{Cl}_k(G) = K_n \), then \( G \) satisfies Property \( P \),” which is equivalent to “\( K_n \) satisfies Property \( P \), and \( \text{cs}(P) \leq k \).” Such statements allow us, in Section 4, to compare, for some properties satisfied by \( K_n \), the stability \( s(P) \) with the complete stability \( \text{cs}(P) \).

2.ORE CONDITION AND COMPLETE CLOSURE NUMBER

**Theorem 2.1.** Let \( G \) be a graph of order \( n \geq 4 \) satisfying for some integer \( k, 1 \leq k \leq n \), the property \( P_k: d_G(u) + d_G(\nu) \geq k \) for each nonedge \( uv \) of \( G \). Then \( \text{Cl}_{k+1}(G) \) is complete (i.e., \( cc(G) \geq k + 1 \)) or \( G \) has one of the following two forms:

(i) \( k \geq n - 2 \) and \( G \) is isomorphic to \( K_{1+2n-2} + (K_r \cup K_{2n-r-1}) \) or \( K_{k+2n-k-3} + (K_r \cup K_{2n-r-1}) \) for some integer \( r \) with \( 1 \leq r \leq 2n - k - 3 \).

(ii) \( k \) is even and \( G \) is isomorphic to \( A + C \) where \( A \) is any graph of order \( a \) with \( 0 \leq a \leq k/2 \) and \( C \) is any \((k/2 - a)\)-regular graph of order \( n - a \).

**Proof.** Let us suppose that the \((k + 1)\)-closure \( H \) of \( G \) is not complete. Then clearly \( H \) satisfies the property \( P_k \) and more precisely the property \( Q_k: d_H(u) + d_H(\nu) = k \) for each nonedge \( uv \) of \( H \). Notice that for each nonedge \( uv \) of \( H \), \( u \), and \( v \) have the same neighbors in \( G \) as in \( H \).

We first study the structure of \( H \) and consider the partition \( V = A \cup B \cup C \) with \( A = \{x \in V; d_H(x) > (k/2)\} \), \( B = \{x \in V; d_H(x) < (k/2)\} \), and in the case \( k \) is even, \( C = \{x \in V; d_H(x) = (k/2)\} \). By \( Q_k \), both \( A \) and \( B \) induce cliques in \( H \), and all the edges between \( A \) and \( C \) and between \( B \) and \( C \) are in \( H \). Let us denote by \( a, b, \) and \( c \) the respective cardinalities of \( A, B, \) and \( C \).

1. Case \( C = \emptyset \) \((k \text { odd or even})\).

The graph \( H \) contains the spanning subgraph \( K_a \cup K_b \) and necessarily \( a > n/2 > b \). If \( k \leq n - 3 \), then \( \text{Cl}_{k+1}(K_a \cup K_{n-a}) = K_n \) and thus \( H = K_n \), a contradiction. Therefore \( k \geq n - 2 \).

1a. Subcase \( k = n - 2 \).
If $H$ contains an edge $uv$ with $u \in A$ and $v \in B$, then for each $w$ in $B - v$ we have $d_H(u) + d_H(w) \geq a + (n - a - 1) = k + 1$ and $uw$ is an edge of $H$. Similarly, $H$ contains each edge $vt$ with $t \in A$, and hence $H$ is complete, in contradiction to our assumption. Thus $H = K_a \cup K_{n-a}$.

1b. Subcase $k = n - 1$.

If $B = \{v\}$, then by $P_k$ the vertex $v$ has some neighbor $u$ in $A$, and exactly one such neighbor, for otherwise $H$ would be complete. So $H = K_1 + (K_1 \cup K_{n-2})$. Thus we now suppose $|A| > |B| \geq 2$.

If there exists a vertex $u$ in $A$ with $d_H(u) \geq a + 1$, that is, $u$ has at least two neighbors $v_1$ and $v_2$ in $B$, then for each vertex $v$ of $B$ different from $v_1$ and $v_2$, $d_H(u) + d_H(v) \geq (a + 1) + (n - a - 1) = k + 1$, and so every vertex of $B$ is adjacent to $u$. In this case, let us consider the partition $V = R \cup S \cup B$ where $S$ is the set of vertices of degree $n - 1$ in $H$. The integers $r = |R|$ and $s = |S|$ satisfy $r \geq 1$ and $1 \leq s \leq n - r - 2$. For any nonedge $uv$ of $H$ with $u \in R$ and $v \in B$, we have $d_H(u) + d_H(v) \geq (r + s - 1) + (n - r - 1) = n + s - 2$ and by $Q_k$, $d_H(u) + d_H(v) = n - 1$. So necessarily $s = 1$, $d_H(u) = r$ for each $u$ in $R$ and $d_H(v) = n - r - 1$ for each $v$ in $B$. Therefore $H = K_1 + (K_1 \cup K_{n-r-1})$.

So we can suppose that the edges of $H$ between $A$ and $B$ form a matching. This matching contains at least two edges $uv$ and $u'v'$ for otherwise there would exist $x$ in $A$ and $y$ in $B$ with $d_H(x) = a - 1$ and $d_H(y) = n - a - 1$ in contradiction with $P_k$. But then $d_H(u) + d_H(v') = a + (n - a) = k + 1$ in contradiction with $Q_k$. This configuration is therefore impossible.

1c. Subcase $k = n$.

If $B = \{v\}$, it is clear by applying Property $Q_k$ to a nonedge $uv$ of $H$ that $d_H(v) = 2$ and then $H = K_2 + (K_1 \cup K_{n-3})$. Assume now $|A| > |B| \geq 2$.

Suppose first that some vertex has degree $n - 1$ in $H$ and consider the partition $V = R \cup S \cup B$ as in the case $k = n - 1$. For every nonedge $uv$ of $H$ with $u \in R$ and $v \in B$ we have $d_H(u) \geq r + s - 1$, $d_H(v) \geq n - r - 1$ and by $Q_k$, $n \geq (r + s - 1) + (n - r - 1) = n + s - 2$. So $1 \leq s \leq 2$.

If $s = 2$ then $d_H(u) = r + s - 1$, $d_H(v) = n - r - 1$, $u$ has no neighbor in $B$, $v$ has no neighbor in $R$ and there is no edge between $R$ and $B$. Thus $H = K_2 + (K_2 \cup K_{n-r-2})$.

If $s = 1$, either $d_H(u) = r + 1$ and $d_H(v) = n - r - 1$, or $d_H(u) = r$ and $d_H(v) = n - r$. Suppose that $d_H(u) = r + 1$ and $d_H(v) = n - r - 1$. This implies that $u$ has in $H$ exactly one neighbor $v'$ in $B$ and $v$ has no neighbor in $R$. Since the vertex $v'$ is not in $S$, it admits at least one nonneighbor $u'$ in $R$. For the nonedge $u'v'$ we have $d_H(v') = n - r$ and $d_H(u') = r$, which means that $u'$ has no neighbor in $B$. Therefore the
nonedge \( u'v \) contradicts \( Q_k \). The same argument holds if \( d_H(u) = r \) and thus the case \( s = 1 \) is impossible.

We now suppose that no vertex has degree \( n - 1 \) in \( H \). If \( d_H(v) = n - a - 1 \) for some vertex \( v \) of \( B \), then by \( P_k \), \( d_H(u) \geq a + 1 \) for each vertex \( u \) of \( A \). Let \( w \) be a vertex of \( B \) of degree at least \( n - a \) and \( u \) a vertex of \( A \) nonadjacent to \( w \). These vertices satisfy \( d_H(u) + d_H(v) \geq (a + 1) + (n - a) = k + 1 \) which is impossible.

Thus each vertex of \( B \) has at least one neighbor in \( A \) and similarly each vertex of \( A \) has at least one neighbor in \( B \). Since \( |A| > |B| \), there exists a vertex \( v \) of \( B \) having at least two neighbors in \( A \). Let \( u \) be a vertex of \( A \) nonadjacent to \( v \). We have \( d_H(u) + d_H(v) \geq a + (n - a + 1) = k + 1 \), which is again impossible.

Therefore if \( C = \emptyset \), then \( k = n - 2 \), \( n - 1 \), or \( n \), and \( H \) is isomorphic to \( K_{k+2-n} + (K_r \cup K_{2n-k-2-r}) \) for some \( r \) between 1 and \( 2n - k - 3 \). We now have to come back to \( G \). Let \( uv \) be a nonedge of \( H \), which happens for every \( u \in K_r \), every \( v \in K_{2n-k-2-r} \) and implies \( d_H(u) + d_H(v) = k \). Then \( uv \) is also a nonedge of \( G \) and, from property \( P_k \), satisfies \( d_G(u) + d_G(v) = k \). This is possible only if \( u \) and \( v \) have the same neighbors in \( G \) as in \( H \), whence \( G = H \) except perhaps in the case when \( k = n \). In this last case, the edge of the clique \( K_{k+2-n} = K_2 \) of \( H \) may not exist in \( G \).

2. Case \( C \neq \emptyset \) (\( k \) even).

By considering the degree of the vertices of \( C \), we see that \( a + b \leq k/2 \) and thus \( c \geq k/2 \). Therefore \( B = \emptyset \) for otherwise every vertex of \( B \) would have degree at least \( k/2 \), and \( H \) is isomorphic to \( K_n + C \) where \( C \) is a \((k/2 - a)\)-regular graph of order \( n - a \) for some \( a \) with \( 0 \leq a \leq k/2 \). Since \( k/2 < n - 1 \), every vertex in \( C \) is the endvertex of a nonedge in \( H \), whence has the same neighbors in \( G \) as in \( H \). However any edge of \( H \) between two vertices of \( A \) may not exist in \( G \).

Let us remark that the theorem is no longer true for \( k = n + 1 \) as shown for instance by the graph consisting of two cliques \( A = K_{2n-3} \) and \( B = K_{n-3} \) (with \( n \equiv 0 \mod 3 \)), where each vertex of \( A \) has exactly one neighbor in \( B \) and each vertex of \( B \) exactly two neighbors in \( A \).

When \( k \) is odd, the statement of Theorem 1 can be simplified:

**Corollary 2.2.** Let \( G \) be a graph of order \( n \geq 4 \) and \( k \) an odd integer with \( 1 \leq k \leq n \). If \( d_G(u) + d_G(v) \geq k \) for each nonedge \( uv \) of \( G \) (Property \( P_k \)), then \( C_{k+1}(G) = K_n \) or \( k \equiv n - 2 \mod 4 \) and \( G \) is isomorphic to \( K_{k+2-n} \cup (K_r \cup K_{2n-k-2-r}) \) or to \( K_{k+2-n} \cup (K_r \cup K_{2n-k-2-r}) \) for some integer \( r \) with \( 1 \leq r \leq 2n - k - 3 \).

The following corollary contains partial results of some Bondy’s or Schmeichel-Hayes’s theorems [2,8].
Corollary 2.3. Every graph $G$ of odd order $n$ is pancyclic if the degree sum of each pair of nonadjacent vertices is at least $n$.

2. Every graph $G$ of even order $n$ is hamiltonian or isomorphic to $K_1 + (K_r \cup K_{n-r-1})$ for some integer $1 \leq r \leq n - 2$ if the degree sum of each pair of nonadjacent vertices is at least $n - 1$.

3. Every graph $G$ of odd order $n$ such that the degree sum of each pair of nonadjacent vertices is at least $n - 2$ admits a hamiltonian path or is the disjoint union of two cliques.

**Proof.**

1. By Theorem 2.1, either $Cl_{n+1}(G)$ is complete and we will see (Theorem 4.1) that this implies that $G$ is pancyclic, or $G$ is isomorphic to $K_2 + (K_r \cup K_{n-2-r})$ or to $K_2 + (K_r \cup K_{n-2-r})$ for some integer $r$ with $1 \leq r \leq n - 3$ and thus still pancyclic.

2. If $G \neq K_1 + (K_r \cup K_{n-r-1})$ then $Cl_n(G)$ is complete and $G$ is hamiltonian.

3. If $G$ is not the disjoint union of two cliques, then $Cl_{n-1}(G)$ is complete and $G$ admits a hamiltonian path.

### 3. NECESSARY CONDITIONS FOR A GRAPH TO HAVE ITS $k$-CLOSURE COMPLETE

In this section we prove that if the $k$-closure of a graph $G$ of order $n$ is complete, then $G$ satisfies some given properties.

The first result concerns the minimum number of edges and is due to L. Clark, R.C. Entringer, and D.E. Jackson [6].

**Theorem 3.1.** For $k \leq 2n - 2$, the size $m$ of a graph $G$ of order $n$ such that $Cl_k(G) = K_n$ satisfies

$$
 m \geq \begin{cases} 
 (k^2 + 4k + 3)/8, & k \equiv 1 \pmod{2}; \\
 (k^2 + 4k + 4)/8, & k \equiv 2 \pmod{4}; \\
 (k^2 + 4k)/8, & k \equiv 0 \pmod{2}.
\end{cases}
$$

In the same paper, it is shown that this bound on $m$ is best possible.

From Theorem 3.1 and classical results in extremal graph theory, we deduce some configurations that are necessarily contained in the graphs whose $k$-closure is complete. But the results so obtained are generally not very good, and it is preferable, if possible, to find direct proofs as in Corollary 3.6 and in Theorem 3.7.

**Corollary 3.2.** If $Cl_k(G) = K_n$ for some $k \geq 2\sqrt{2(t - 2)n}$, where $t$ is an integer greater than 2 and $n \geq 2t - 3$, then $G$ contains any tree $T_t$ on $t$ vertices and in particular the path $P_t$. 
Proof. It is well known that if $G$ is a graph or order $n \geq 2t - 3$ and size at least $(t - 2)n$, then $G$ contains any tree on $t$ vertices. Let us shortly recall the proof. The result is true for a graph with $2t - 3$ vertices since such a graph is complete. Proceeding by induction and assuming that the property is true for $n - 1$ vertices, we also get that it is true for $n$ vertices since in a graph $G_n$ of order $n$, either there is a vertex $x$ of degree at most $t - 2$, and then $G_n - x$ satisfies the induction hypothesis, or every vertex in $G_n$ has degree at least $t - 1$ and the result follows directly.

If $G$ satisfies the hypothesis of Corollary 3.2, then from Theorem 3.1, $G$ has at least $(k^2 + 4k)/8 > (t - 2)n$ edges, whence $G$ contains any tree $T_t$ on $t$ vertices.

Corollary 3.3. If $\text{Cl}_k(G) = K_n$ for some $k = 20\sqrt{2}/ln^{(1+1/2l)}t$ for some integer $l$, then $G$ contains an even cycle $C_{2p}$ for every integer $p \in [l, ln^{1/2l}]$.

Proof. In [5], Bondy and Simonovits have proved that every graph with order $n$ and size greater than $100ln^{1+1/l}t$ contains an even cycle $C_{2p}$ for every $p \in [l, ln^{1/2l}]$. If $G$ satisfies the hypothesis of Corollary 3.3, then from Theorem 3.1, $G$ has at least $(k^2 + 4k)/8 > 100ln^{1+1/l}t$ edges, whence $G$ has an even cycle $C_{2p}$ for every $p \in [l, ln^{1/2l}]$.

Corollary 3.4. If $\text{Cl}_k(G) = K_n$ for some $k = 2\sqrt{2}/n^{1-(1/2t)}$ for some integers $t$ and $s$ with $2 \leq t \leq s \leq n - t$, then $G$ contains a complete bipartite graph $K_{s,t}$.

Proof. It is a consequence of a result of Bollobás [1] establishing that if a graph $G$ of order $n$ contains no $K_{t,s}$, $2 \leq t, 2 \leq s$, then $G$ has at most $\frac{1}{2}(s - 1)^{1/t}(n - t - 1)n^{1-1/t} + \frac{1}{2}(t - 1)n$ edges. From Theorem 3.1, we obtain that a graph $G$ such that $\text{Cl}_k(G) = K_n$ with $k = 2\sqrt{2}/n^{1-(1/2t)}$ has more than $\frac{1}{2}(s - 1)^{1/t}(n - t - 1)n^{1-1/t} + \frac{1}{2}(t - 1)n$ edges, whence the wanted result.

In the particular case when $t = 2$, we can obtain a direct and better result as a corollary of the following theorem.

Theorem 3.5. Let $G$ be a graph on $n$ vertices with $\text{Cl}_k(G) = K_n$ for some $k \leq 2n - 3$. Define the sequence of integers $d_1 \geq d_2 \geq \cdots \geq d_n$ and the sequence of vertices $x_1, x_2, \ldots, x_n$ as follows

1. $d_i = \max\{d(v) \mid v \in V(G)\}$ and $x_1 \in V(G)$ with $d(x_1) = d_1$;
2. For $1 \leq i \leq n - 1$, $H_i = G - \{x_1, \ldots, x_i\}$, $d_{i+1} = \max\{d_{H_i}(v) \mid v \in V(H_i)\}$ and $x_{i+1} \in V(H_i)$ with $d_{H_i}(x_{i+1}) = d_{i+1}$.

Then $d_i + i - 1 \geq \lceil k/2 \rceil (1 \leq i \leq n)$. 


Proof. Let \( d_i^* \geq d_i^* \geq \cdots \geq d_n^* \) and \( x_1, x_2, \ldots, x_n \) be defined as above, and suppose there exists an \( i \geq 1 \) with \( d_i^* + i - 1 < \lceil k/2 \rceil \). Let \( t \) be the least integer such that \( d_i^* + t - 1 < \lceil k/2 \rceil \), hence \( d_i^* + t - 1 \leq \lceil k/2 \rceil - 1 \leq (k - 1)/2 \). For each \( i \in \{t, t + 1, \ldots, n\} \) we have \( d_i^* \leq d_i^* \leq (k - 1)/2 - (t - 1) \). Set \( X = \{x_1, x_{t+1}, \ldots, x_n\} \). In the construction of the \( k \)-closure of \( G \), assume that \( uv \) is the first edge that we want to add between two vertices of \( X \). Even if we have already added every edge between \( u \) and \( V(G) - X \), \( d(u) \leq (k - 1)/2 - (t - 1) + |V(G) - X| \leq (k - 1)/2 \). Similarly, \( d(v) \leq (k - 1)/2 \), so it is impossible to add any more edges between two vertices of \( X \), and since \( \text{Cl}_k(G) = K_n \), the set \( X \) induces in \( G \) a complete subgraph. This means \( |X| - 1 \geq (n - (t - 1)) - 1 = n - t \). But since \( d_i^* \leq (k - 1)/2 - (t - 1) \), this would give \( k \geq 2n - 1 \), contradicting \( k \leq 2n - 3 \).

We notice that Theorem 3.5 implies that if \( \text{Cl}_k(G) = K_n \) with \( k \leq 2n - 3 \), then \( G \) contains at least \( \lceil k/4 \rceil \) vertices of degree at least \( \lceil k/4 \rceil \).

**Corollary 3.6.** If \( \text{Cl}_k(G) = K_n \) with \( k \geq \sqrt{8s(n - 1)} \) for some integer \( s \) such that \( 1 \leq s \leq n - 2 \), then \( G \) contains a complete bipartite graph \( K_{2s} \).

**Proof.** Suppose \( \text{Cl}_k(G) \) is complete and \( G \) contains no \( K_{2s} \). This implies that any two vertices have at most \( s - 1 \) common neighbors. Set \( q = \lceil k/2 \rceil + 1 \) and let \( d_i^* \geq d_2^* \geq \cdots \geq d_n^* \) and \( x_1, x_2, \ldots, x_n \) be defined as in Theorem 3.5. Then we know that \( d(x_i) \geq d_i^* \geq \lceil k/2 \rceil - (i - 1) \). The consideration of the successive neighborhoods of \( x_1, \ldots, x_q \) gives

\[
n \geq (d(x_1) + 1) + (d(x_2) - (s - 1)) + \cdots + (d(x_q) - (q - 1) (s - 1))
\]

\[
\geq 1 + \left\lfloor \frac{k}{2} \right\rfloor + \left( \left\lfloor \frac{k}{2} \right\rfloor - 1 - (s - 1) \right) + \cdots + \left( \left\lfloor \frac{k}{2} \right\rfloor - (q - 1) - (s - 1) \right)
\]

\[
= 1 + q \left\lfloor \frac{k}{2} \right\rfloor - s(1 + 2 + \cdots + (q - 1))
\]

\[
= 1 + q \left\lfloor \frac{k}{2} \right\rfloor - sq(q - 1)/2.
\]

but

\[
s(q - 1) \leq \left\lfloor \frac{k}{2} \right\rfloor \text{ and } q > \frac{\lceil k/2 \rceil}{s}.
\]
So

\[
n \geq 1 + q \left[ \frac{k}{2} \right] - \frac{q[k/2]}{2} > 1 + \frac{[k/2]^2}{2s} \\
\geq 1 + \frac{k^2}{8s} \geq 1 + \frac{8s(n-1)}{8s} = n,
\]

a contradiction.

Similarly, the following result related to the existence of cliques is better than the one that could be obtained from Theorem 3.1 and Turán's theorem.

**Theorem 3.7.** Let \( t \) be a positive integer and \( G \) a graph of order \( n \geq t \). If \( C_l(t, t, r) = K_n \) with \( k(n, t) = 2\left(\frac{(t - 2)}{(t - 1)}n\right) + 1 \), then \( G \) contains a clique \( K_t \). This value \( k(n, t) \) is best possible.

**Proof.** Let us begin with a preliminary arithmetical remark: if \( n \) and \( t \) are two integers with \( n \geq t \geq 3 \) and if \( a = \left(\frac{(t - 2)}{(t - 1)}n\right) + 1 \), then 

\[
\left(\frac{(t - 3)}{(t - 2)}a\right) - n - a \leq \left(\frac{(t - 2)}{(t - 1)}n\right). \quad \text{Indeed,} \quad \left(\frac{(t - 3)}{(t - 2)}a\right) - a \leq \frac{(a)}{(t - 2)} \quad \text{and} \quad a > \left(\frac{(t - 2)}{(t - 1)}n\right).
\]

Therefore

\[
\left(\frac{(t - 3)}{(t - 2)}a\right) + n - a < \left(\frac{(t - 2)}{(t - 1)}n\right), \quad \text{which is the required inequality since the first member is integer.}
\]

We now prove the theorem by induction on \( t \). For \( t = 2 \), the condition \( C_l(2, 3, r) = K_n \) means that \( G \) contains at least one edge and the property is true. Suppose it is true for \( 2 \leq t < T \) and let \( G \) be a graph of order \( n \geq T \) such that \( C_l(n, T) = K_n \). This graph \( G \) contains a vertex \( x \) of degree at least \( \left(\frac{(T - 2)}{(T - 1)}n\right) + 1 \). Let \( A \) be a set of \( a = \left(\frac{(T - 2)}{(T - 1)}n\right) + 1 \) neighbors of \( x \) (note that \( a \geq T - 1 \) since \( n \geq T \) and \( \left(\frac{(T - 2)}{(T - 1)}n\right) = T - 2 \)).

If the closure \( A' = C_l(T, T - 1)(A) \) is not complete, then there exist nonadjacent vertices in \( A' \) and for every pair \((z_1, z_2)\) of such vertices, \( d_{\ell}(z_1) + d_{\ell}(z_2) \leq 2\left(\frac{(T - 3)}{(T - 2)}a\right) \). In the construction of the \( k(n, T) \)-closure of \( G \), we may perhaps add, in addition to all the edges of \( A' \) that are not in \( A \), all the edges between \( A \) and \( V - A \). But even after these eventual additions, the degree sum \( d(z_1) + d(z_2) \) remains upperbounded by \( 2\left(\frac{(T - 3)}{(T - 2)}a\right) + 2|V - A| \), which is at most \( 2\left(\frac{(T - 2)}{(T - 1)}n\right) \) by the preliminary remark. Thus in this case the \( k(T, n) \)-closure of \( G \) is not complete, in contradiction with the hypothesis on \( G \).

Therefore \( C_l(t, T - 1)(A) = K_n \) and by the induction hypothesis, \( A \) contains a clique \( K_{T - 1} \), with forms with \( x \) a clique \( K_T \) of \( G \). The property is thus true for each value of \( t \). The value of \( k(n, t) \) is best possible as shown by the complete \((t - 1)\)-partite graph containing \( r \) independent sets of order \( q + 1 \) and \( t - 1 - r \) independent sets of order \( q \), where \( n = q(t - 1) + r \) with \( 0 \leq r < t - 1 \). The minimum degree of this graph is \( \left(\frac{(t - 2)}{(t - 1)}n\right) \)
and thus its \((k(n, t) - 1)\)-closure is complete. However, it does not contain any clique \(K_t\).

### 4. THE COMPLETE STABILITY OF SOME PROPERTIES

We present here in terms of the complete stability some results obtained in the previous sections and mention some other ones. As throughout this paper, we consider graphs of order \(n\) and properties that are satisfied by \(K_n\) (in some cases only for \(n\) sufficiently large).

For the properties that are related to the existence of cycles, paths, or cliques in \(G\), we generally obtain better (that is smaller) values for \(cs(P)\) than for \(s(P)\). The first example, which motivated us, is Theorem 4.1, which implies that for any integer \(t\) between 3 and \(n\), the property \(P\): “\(G\) contains a cycle \(C_t\)” has \(cs(P) \leq n + 1\). In Theorems 4.2–4.4, the bounds on \(cs(P)\) are improved for some particular values of \(t\) or \(n\).

**Theorem 4.1.** The complete stability of the property \(P\): “\(G\) is pancyclic” satisfies \(cs(P) = n + 1\) if \(n\) is even and \(n \leq cs(P) \leq n + 1\) if \(n\) is odd.

**Proof.** If the \((n + 1)\)-closure of a graph of \(G\) of order \(n\) is complete, then by [3], \(G - x\) contains a hamiltonian cycle \(C(x)\) for every vertex \(x\) of \(G\). Let us choose a vertex \(x\) of maximum degree \(\Delta\) and consider the neighbors \(x_1, x_2, \ldots, x_\Delta\) of \(x\) on \(C(x)\). If \(G\) contains no cycle \(C_l\) for some \(l\) with \(3 \leq l \leq n\), then no vertex \(y_i\), \(1 \leq i \leq \Delta\), is adjacent to \(x\), where \(y_i\) is the vertex at distance \(l - 2\) from \(x_i\) on the arbitrarily oriented cycle \(C(x)\). Thus \(2\Delta \leq n - 1\), in contradiction with \(C_{n+1}(G) = K_n\). Therefore \(G\) is pancyclic and \(cs(P) \leq n + 1\). Moreover, if \(n\) is even, the complete bipartite graph \(K_{n2,n2}\) has \(n\)-closure complete and contains no odd cycles. So \(cs(P) > n\). If \(n\) is odd, the complete bipartite graph \(K_{(n+1)2,(n-1)2}\) has \((n - 1)\)-closure complete and contains no odd cycles. So \(cs(P) > n - 1\).

**Theorem 4.2.** The property \(P\): “\(G\) is hamiltonian” satisfies \(cs(P) = n\).

**Proof.** We know by [3] that \(cs(P) \leq s(P) \leq n\). On the other hand, the graph \(G = K_1 + (K_{(n-2)2} \cup K_{n2})\) if \(n\) is even, or \(G = K_1 + (K_{(n+1)2} \cup K_{(n-1)2})\) if \(n\) is odd, has its \((n - 1)\)-closure complete and is not hamiltonian. So \(cs(P) > n - 1\).

**Theorem 4.3.** For every integer \(r\) between 1 and \([n/2] - 1\), the property \(P\): “\(G\) contains a cycle \(C_{2r+1}\)” satisfies \(cs(P) = n + 1\) if \(n\) is even and \(n \leq cs(P) \leq n + 1\) if \(n\) is odd.
**Proof.** We know that \( cs(P) \equiv n + 1 \) from Theorem 4.1. Moreover, the graphs \( K_{n/2, n/2} \) if \( n \) is even and \( K_{(n+1)/2, (n-1)/2} \) if \( n \) is odd give a lower bound for \( cs(P) \).

**Theorem 4.4.** When \( n \) is odd the property \( P \): "\( G \) contains a cycle \( C_t \)" satisfies \( cs(P) \leq n \) for each integer \( t \) between 3 and \((n + 19)/13\). Moreover, if \( t \) is odd then \( cs(P) = n \).

**Proof.** It is shown in [7] that if \( G \) is a hamiltonian graph of odd order \( n \) containing two nonadjacent vertices \( x \) and \( y \) such that \( d(x) + d(y) \geq n \), then \( G \) contains cycles of all lengths between 3 and \((n + 19)/13\). If the \( n \)-closure of \( G \) is complete, the hypothesis conditions are satisfied and so \( cs(P) \leq n \). If, moreover, \( t \) is odd, \( cs(P) = n \) by Theorem 4.3.

For the property of containing the cycle \( C_3 \), the value \( 2[n/2] + 1 \) of \( cs(P) \) is a particular case of the following result on cliques which is a direct consequence of Theorem 3.7.

**Theorem 4.5.** The property \( P \): "\( G \) contains a clique \( K_t \)" satisfies \( cs(P) = 2[\left((t - 2)/(t - 1)\right)n] + 1 \).

For the properties in the following list, the stability \( s(P) \) is determined in [3]. For each of them, the example of the graph showing that \( s(P) \) is best possible has its \( (s(P) - 1) \)-closure complete and does not satisfy \( P \). Therefore for all of these properties, \( cs(P) = s(P) \).

For \( P \): "\( G \) is \( s \)-hamiltonian (or \( s \)-edge-hamiltonian)"\), \( cs(P) = n + s \).
Limit example: \( K_{s+1} + (K_{n-s-2} \cup K_1) \).

For \( P \): "\( G \) is \( s \)-hamilton-connected, \( cs(P) = n + s + 1 \).
Limit example: \( K_{s+2} + (K_{n-s-3} \cup K_1) \).

For \( P \): "\( G \) is \( s \)-connected (or \( s \)-edge-connected)"\), \( cs(P) = n + s - 2 \).
Limit example: \( K_{s-1} + (K_{n-s} \cup K_1) \).

For \( P \): "\( \alpha(G) \leq s \)"\), \( cs(P) = 2n - 2s - 1 \).
Limit example: \( K_{n-s-1} + \overline{K}_{s+1} \).

For \( P \): "\( \mu(G) \leq s \)"\), \( cs(P) = n - s \).
Limit example: \( K_{n-s} + \overline{K}_s \).
For $P$: "$G$ contains a matching of $s$ edges with $s \leq n/2$", $\text{cs}(P) = 2s - 1$.

Limit example: $K_{s-1} + \overline{K_{n-s+1}}$.

For the following six properties, we only give an interval containing $\text{cs}(P)$ and not its exact value.

**Theorem 4.6.** For $2 \leq s \leq n$, the property $P$: "$G$ admits a $s$-factor" satisfies $n + s - 2 \leq \text{cs}(P) \leq n + 2s - 4$.

**Proof.** Recall that a $s$-factor is a $s$-regular spanning subgraph. We know from [3] that $s(P) \leq n + 2s - 4$ (and that $s(P) = n + 2s - 4$ if $s \leq (n/3) - 1$). On the other hand, the graph $K_{s-1} + (K_{n-s} \cup K_1)$ has its $(n + s - 3)$-closure complete and contains no $s$-factor. So $\text{cs}(P) > n + s - 3$.

Note that for the property of containing a 2-factor, we obtain $\text{cs}(P) = s(P) = n$ as for a hamiltonian cycle.

**Theorem 4.7.** The property $P$: "$G$ contains two edge-disjoint hamiltonian cycles" satisfies $n + 2 \leq \text{cs}(P) \leq n + 4$.

**Proof.** Suppose that $\text{Cl}_{n+4}(G) = K_n$. Then $G$ contains a hamiltonian cycle $C$ and the partial graph $H = G - E(C)$ satisfies $d_H(x) = d_G(x) - 2$ for each vertex. Therefore $\text{Cl}_{n}(H) = K_n$ and $H$ contains another hamiltonian cycle $C'$ that is edge-disjoint from $C$. Thus $\text{cs}(P) \leq n + 4$. On the other hand, the graph $K_3 + (K_{n-4} \cup K_1)$ satisfies $\text{Cl}_{n+1}(G) = K_n$ but not Property $P$. Therefore $\text{cs}(P) > n + 1$.

**Theorem 4.8.** The property $P$: "$G$ is hamiltonian and for every hamiltonian cycle $C$, there exists another hamiltonian cycle $C'$ edge-disjoint from $C"$, satisfies $n + 3 \leq \text{cs}(P) \leq n + 4$.

**Proof.** The previous proof shows that $\text{cs}(P) \leq n + 4$. On the other hand, let us construct the graph $G$ from $K_{n-7} + K_4$ where the vertices of $K_4$ are $x_1, x_2, x_3, x_4$, by adding a 3-path $uvw$ and the edges from $u$ to $x_1, x_2, x_3$, from $w$ to $x_2, x_3, x_4$ and from $v$ to $x_1$ and $x_4$. This graph has its $(n + 2)$-closure complete and does not satisfy $P$ if we start by a hamiltonian cycle through $x_1 uvwx_4$. Therefore $\text{cs}(P) > n + 2$.

**Theorem 4.9.** For $2 \leq t \leq (n + 3)/2$, the property $P$: "$G$ contains every tree on $t$ vertices" satisfies $\sqrt{(t - 2)n}/\sqrt{2} \leq \text{cs}(P) \leq 2\sqrt{2(t - 2)n}$.
Proof. The upper bound is a direct consequence of Corollary 3.2 in the previous section and the lower bound is provided by the following graph $G_0$. Assume that $t$ is even, $t = 2m \leq (n + 3)/2$, $G_0$ consists of \[\left\lfloor n/(\sqrt{(m-1)n} + m - 1) \right\rfloor\] bipartite complete graphs $K_{m-1, \sqrt{(m-1)n}}$ and \[n - \left\lfloor n/(\sqrt{(m-1)n} + m - 1) \right\rfloor \left(\sqrt{(m-1)n} + m - 1\right)\] isolated vertices. Clearly $G_0$ does not contain balanced trees on $t$ vertices (i.e., trees with bipartition $(m,m)$). However, $\text{Cl}_{\sqrt{(t-2)n}/2}(G_0) = K_n$. Indeed, in the closure process, first we join each vertex of each $(m-1)$-independent set to all the other vertices since its degree in $G$ is exactly $\sqrt{(m-1)n} = \sqrt{(t-2)n}/\sqrt{2}$. Then, we join together all the vertices belonging to all the $\sqrt{(m-1)n}$-independent sets. Finally, each isolated (in $G$) vertex is joined to each vertex that is not yet in its neighborhood and we get a complete graph. \[\square\]

Remark 4.10. The property $P$ : “$G$ contains a $P_4$ (i.e., a path with 4 vertices)” satisfies $\sqrt{8n + 9} - 3 \leq \text{cs}(P) \leq \sqrt{8n} + 26 - 3$.

Proof. Suppose $\text{Cl}_k(G)$ is complete for some $k \geq \sqrt{8n + 26} - 3$, but $G$ contains no $P_4$. Let $d_1^* \geq d_2^* \geq \cdots \geq d_n^*$ and $x_1, x_2, \ldots, x_n$ be defined as in Theorem 3.5. If $d_i^* \geq 2 \ (i \neq j)$, then, since $G$ contains no $P_4$, $x_i \notin N(x_j)$, $x_j \notin N(x_i)$, and $N(x_i) \cap N(x_j) = \emptyset$ ($N(x)$ denotes the set of neighbors of $x$). Since $d_i^* \geq [k/2] - (i - 1)$, the sets $\{x_i \cup N(x_i)\}$ form disjoint sets in $V(G)$ for $1 \leq i \leq [k/2] - 1$. So we have

\[
\begin{align*}
\sum_{i=1}^{[k/2]-1} (1 + d(x_i)) &\geq \sum_{i=1}^{[k/2]-1} (1 + [k/2] - (i - 1)) \\
&= \frac{1}{2} \frac{([k/2] + 1) ([k/2] + 2)}{2} - 3 \\
&= \frac{1}{2} \frac{((k + 2)/2) + 1) ((k + 4)/2) + 2)}{2} - 3 \\
&\geq \frac{8n + 26 - 1}{8} - 3 \\
&= \frac{8n + 26 - 1 - 24}{8} > n,
\end{align*}
\]

a contradiction. This means $\text{cs}(P) \leq \sqrt{8n} + 26 - 3$.

The lower bound is provided by the graph $G_1$ consisting of $d$ stars with, respectively, 1, 2, 3, $\ldots$, $d$ edges. It has exactly $n = (d^2 + 3d)/2$ vertices, it does not contain any $P_4$, and it satisfies $\text{Cl}_{2d-1}(G_1) = K_n$, whence the result since $2d - 1 = \sqrt{8n} + 9 - 4$. \[\square\]

Theorem 4.11. The property $P$ : “$G$ contains a complete bipartite graph $K_{2,s}$” satisfies $\sqrt{8n + 9} - 4 \leq \text{cs}(P) \leq \sqrt{8(s-1)n}$. 
Proof. The upper bound is a direct consequence of Corollary 3.6 in the previous section and the lower bound is provided by the graph $G_1$, which is defined above.

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