On Ramsey Numbers Involving Starlike Multipartite Graphs

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ABSTRACT

The Ramsey number \( r(G, H) \) is evaluated exactly in certain cases in which both \( G \) and \( H \) are complete multipartite graphs \( K(n_1, n_2, \ldots, n_k) \). Specifically, each of the following cases is handled whenever \( n \) is sufficiently large: \( r(K(1, m_1, \ldots, m_r), K(1, n)), r(K(1, m), K(n_1, \ldots, n_k, n)) \), provided \( m \geq 4 \), and \( r(K(1, 1, m), K(n_1, \ldots, n_k, n)) \).

1. INTRODUCTION

If \( G \) and \( H \) are (simple) graphs, define the Ramsey number \( r(G, H) \) to be the smallest \( n \) such that if the edges of the complete graph \( K_n \) are colored red and blue, either the red subgraph contains a copy of \( G \) or the blue subgraph contains a copy of \( H \). An interesting case is that in which \( G \) and \( H \) are both complete multipartite graphs \( K(n_1, \ldots, n_k) \); for convenience, we will write these graphs as \( K(n_1, \ldots, n_k) \). There is no real hope of evaluating these numbers in general, since this case includes the very difficult classical case in which \( G \) and \( H \) are complete graphs.

Indeed, there seems to be little hope for exact evaluations unless, in some sense, each of \( G \) and \( H \) is small or sparse. In this context, "sparse" means that one part of the complete multipartite graph is much larger than the others, and often that the smallest part has just one vertex. We may think of such graphs as being "starlike." The simplest and sparsest multipartite graphs are, of course, the stars \( K(1, n) \), and \( r(K(1, m), K(1, n)) \).
has been completely evaluated [7]. Here we will prove the following three main theorems, in which at least one graph is more general than a star. (In what follows, lower-case italic letters always denote positive integers.)

**Theorem 1.** Let \( m_1 \leq m_2 \leq \cdots \leq m_k \) be fixed. Then if \( n \) is sufficiently large,

\[
r(K(1, m_1, m_2, \ldots, m_k), K(1, n)) = k\cdot(r(K(1, m_1), K(1, n)) - 1) + 1
\]

\[
= \begin{cases} 
  k\cdot(m_1 + n - 2) + 1 & \text{if both } m_i \text{ and } n \text{ are even}, \\
  k\cdot(m_1 + n - 1) + 1 & \text{otherwise}. 
\end{cases}
\]

**Theorem 2.** Let \( m, n_1, n_2, \ldots, n_k \) be fixed, with \( m \geq 4 \) and \( n_i \geq 2 \) for each \( i \). Then if \( n \) is sufficiently large,

\[
r(K(1, m), K(n_1, n_2, \ldots, n_k, n))
\]

\[
= \begin{cases} 
  n + (m - 2) \sum_{i=1}^{k} n_i + k & \text{if } m \text{ is even and } n + k \text{ is odd}, \\
  n + (m - 2) \sum_{i=1}^{k} n_i + k + 1 & \text{otherwise}. 
\end{cases}
\]

**Theorem 3.** Let \( m, n_1, n_2, \ldots, n_k \) be fixed. Then if \( n \) is sufficiently large,

\[
r(K(1, 1, m), K(n_1, n_2, \ldots, n_k, n)) = 2 \left( \sum_{i=1}^{k} n_i + n \right) - 1.
\]

Each of the following three sections will be devoted to the proof of one of these theorems. In Section 3, an extension of Theorem 2 will be indicated in which the \( n_i \) are unrestricted.

### 2. STARLIKE MULTIPARTITE GRAPH VERSUS LARGE STAR

In this section we will prove Theorem 1, but first we need some preliminaries. If \( r(G, H) \leq M \), we write \( K_M \rightarrow (G, H) \). Of course, this notation is often used more generally when \( K_M \) is replaced by an arbitrary graph. In what follows, the words "coloring," "2-coloring," and the like will always refer to colorings of edges, not vertices, by the colors red and blue. (In this paper we will not consider colorings with more than two colors.) If \( K_M \rightarrow (G, H) \), then any 2-coloring of \( K_M \) without a red \( G \) or blue \( H \) will be called a good coloring; the identities of \( G \) and \( H \) will always be clear from context. Naturally, if \( K_M \) has a good coloring,
\( r(G, H) = M + 1 \). In a 2-coloring of a graph \( F \), \((F)_r \) and \((F)_h \) will denote the red and blue subgraphs of \( F \), respectively. Sometimes the parentheses will be dropped.

As indicated in the Introduction, we will write \( K(n_1, \ldots , n_k) \) for \( K_{n_1, \ldots , n_k} \), but we will retain the usual notation \( K_n \) for the complete graph on \( n \) vertices. Also, if \( n_1 = \cdots = n_k = n \), we write \( K(n; k) \) for \( K(n, \ldots , n) \). The \( k \) maximal independent sets of a \( K(n_1, \ldots , n_k) \) will be called its \textit{parts}. These parts therefore have \( n_1, \ldots , n_k \) vertices, respectively.

Terminology and notation not explicitly defined follows Harary [6]. Thus \( V, p, q, \Delta, \delta, g \) represent the set of vertices, the number of vertices, the number of edges, the minimum degree, the maximum degree, and the girth, respectively, of a graph under consideration. These will usually be written \( V(G), p(G), \ldots \), but the \( G \) may be dropped when the context is clear. Also, \( (A) \) denotes the subgraph induced by a subset \( A \) of vertices of a graph, \( kG \) denotes the disjoint union of \( k \) copies of \( G \), and \( \overline{G} \) denotes the complement of \( G \). As usual, if \( \alpha \) is real, then \( \lfloor \alpha \rfloor \) denotes the greatest integer in \( \alpha \). Also, if \( D \) is a set, \( |D| \) is its cardinality.

Our first lemma is already implicit in the statement of Theorem 1.

**Lemma 1.** [7]. For any \( m \) and \( n \),

\[
r(K(1, m), K(1, n)) = \begin{cases} m + n - 1 & \text{if both } m \text{ and } n \text{ are even,} \\ m + n & \text{otherwise.} \end{cases}
\]

Our second lemma will be useful in establishing the lower bound for \( r(G, H) \) in Theorems 1 and 2.

**Lemma 2.** Let \( r \) and \( g \) be fixed. Then for all sufficiently large \( n \) there exists a graph \( G \) with \( p(G) = n \), with girth at least \( g \), and such that the following occurs:

(a) If either \( r \) or \( n \) is even, \( G \) is \( r \)-regular.

(b) If both \( r \) and \( n \) are odd, \( G \) has one vertex of degree \( r - 1 \), and the rest of degree \( r \).

**Proof.** Select a graph \( G \) which is \( r \)-regular and has girth at least \( g \). The existence of such a graph is assured by [5]. Set \( t = \lfloor \frac{r}{2} \rfloor \) and let \( G' \) be the graph \( rG \). Further set \( p = p(tG) = t(p(G)) \); this graph is again \( r \)-regular, and has girth at least \( g \). Now adjoin one new vertex \( x \), delete an edge \( uw \) from each copy of \( G \), and join \( u \) and \( v \) to \( x \) by an edge. This new graph \( G'' \) has \( p + 1 \) vertices and girth at least \( g \). If \( r \) is even, it is also \( r \)-regular. In this case we may clearly construct our desired \( G \) for every \( n \geq p' \) out of disjoint unions of \( G' \) and \( G'' \). If \( r \) is odd, we do the same, but then join vertices corresponding to \( x \) in pairs. If \( n \) is even,
this will produce an r-regular graph. If n is odd, one vertex will have
degree $r - 1$. This completes the proof.

Now we establish the lower bound for Theorem 1.

**Lemma 3.** Let $m_1 \leq m_2 \leq \cdots \leq m_k$ be fixed. Then if $n$ is sufficiently large,

$$f(K(1, m_1, m_2, \ldots, m_k), K(1, n)) \geq k(r(K(1, m_1), K(1, n)) - 1) + 1.$$

**Proof.** Set $r = m_1 - 1, g = 4, M = r(K(1, m_1), K(1, n)) - 1$, and
let $G_M$ be a graph on $M$ vertices with the properties given in Lemma 2.
Two-color $K_{km}$ by letting the blue subgraph be $kG_M$ and the red subgraph
be $k\overline{G}_M$. From the definition of $M$ we see that $G_M$, and hence the blue
subgraph, contains no $K(1, n)$.

Now we must see that the red subgraph contains no $K(1, m_1, \ldots, m_k)$;
in fact, this is the case even when $m_1 = \cdots = m_k = m$. Denote the
vertices of the $k$ copies of $G_M$ in the red subgraph by $A_1, \ldots, A_k$, and
the vertices of the $k + 1$ parts of $K(1, m, \ldots, m)$ by $B_0, \ldots, B_k$, where
$B_0 = 1$. Without loss of generality, we put $B_0$ into $A_1$. We may attempt
to put part of $B_1$, say, into $A_1$, as well, but we cannot put all of it into
$A_1$, since $G_M$ contains no $K(1, m_1) = K(1, m)$. We may not also attempt
to put part of some other $B_i$ into $A_1$, since $G_M$ contains no triangle.
Therefore, without loss of generality, we put $B_0$ and perhaps part of $B_1$
into $A_1$. At least part of the remainder of $B_1$ must then be put into, say,$A_2$. We may then attempt to put some, but not all, of $B_2$ into $A_2$. Proceeding
in this manner we eventually find that part of $B_i$ has no place to go.

Thus we have a good coloring of $K_{km}$, completing the proof.

Our last lemma, extracted from Theorem 2 of [4], establishes a basic
approximation result. In fact, this holds if $K(1, n)$ is replaced by any
tree of the same order.

**Lemma 4.** [4]. If $k$ and $l$ are fixed, then there is a real $\alpha, 0 < \alpha < 1$,
such that for $n$ sufficiently large

$$r(K(l; k), K(1, n)) = (k - 1)n + O(n^\alpha).$$

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** Set $M = k(r(K(1, m_1), K(1, n)) - 1) - 1$. In view
of Lemma 3, we need only show that $K_M \rightarrow (K(1, m_1, \ldots, m_k), K(1, n))$
when $n$ is large. We will assume the contrary, and arrive at a contradiction.
Thus, assume that for some choice of $m_1, \ldots, m_k$ it happens that $K_M$
has a good coloring when $n$ (and hence $M$) is large. (Of course, we must
have $k \geq 2$.) Let $G$ be a good-colored copy of $K_M$ such that
$K(1, m_1, \ldots, m_k) \not\subset (G)_R$ and $K(1, n) \not\subset (G)_R$.

We now apply Lemma 4 to $G$. Since $G$ has $M > kn$ vertices it follows
that for \( l \) fixed, but large compared to \( m_i \), and \( n \) sufficiently large, \( G \) contains a red \( K(l; k) \). Furthermore, since no part of this \( K(l; k) \) can contain a red \( K(1, m_i) \), and since \( l \) is large with respect to \( m_i \), the blue graph induced by each part contains a large complete graph. Hence, by making \( l \) smaller, but still large, we can assume that the good-colored graph \( G \) contains a two-colored \( K_{\ell} \) such that \( (K_\ell)_R = K(l; k) \) and \( (K_\ell)_B = kK_l \). That is, each part in the red \( K(l; k) \) induces a blue complete graph on \( l \) vertices. For convenience label the vertex sets of these \( k \) blue complete graphs \( A_1, A_2, ..., A_k \). Thus each pair of vertices from different \( A_i \) is joined by a red edge and each pair from the same \( A_i \) is joined by a blue edge. Also \( |A_i| = l \) for each \( i \), where \( l \) is large with respect to \( m_i \), but small with respect to \( n \). Set \( A = \cup \cup_{i=1}^k A_i \).

We turn our attention to \( V(G) - A \). For each \( i, 1 \leq i \leq k \), let \( B_i \) be those vertices of \( V(G) - A \) which are adjacent in red to all but at most \( (l - m_i)/(m_i + 1) \) vertices of each \( A_j, j \neq i \). Since \( l \) is large and \( K(1, m_i, ..., m_k) \not\subseteq (G)_R \), these \( B_i \) are pairwise disjoint. Note that for each \( 1 \leq i \leq k \), any set of \( m_i + 1 \) vertices in \( B_i \) has a common red neighborhood of at least \( m_i \) vertices in each \( A_j, j \neq i \). Indeed, the same is therefore true of any set of \( m_i + 1 \) vertices in \( A_i \cup B_i \). Consequently, for each \( 1 \leq i \leq k \), \( (A_i \cup B_i) \) can contain no red \( K(1, m_i) \), for if it did, \( G \) would contain a red \( K(1, m_1, ..., m_k) \).

Now let \( B = \cup_{i=1}^k B_i \), and let \( B_{k+1} = V(G) - A - B \), so that the \( A_i \) and \( B_i \) together partition \( V(G) \). Set \( b_i = |B_i|, 1 \leq i \leq k + 1 \). Our next objective is to determine the magnitude of each \( b_i \) in terms of \( m_i, m_k, n \), and \( l \). Let \( E_i, 1 \leq i \leq k \), be those vertices of \( B_i \) which are adjacent in red to some element of \( A_i \), and let \( E = \cup_{i=1}^k E_i \). Since \( (A_i \cup B_i) \) can contain no red \( K(1, m_i) \), at most \( (m_i - 1)A_i = (m_i - 1)l \) red edges can join \( A_i \) to \( B_i \). Hence \( |E| < m_i l \) and \( |E| < km_i l \).

Turning to \( B_{k+1} \), observe that for any \( v \in B_{k+1} \) (indeed, for any \( v \in V(G) - A \)), there must be an \( i, 1 \leq i \leq k \), for which \( v \) has fewer than \( m_i \) red adjacencies in \( A_i \), for otherwise we would have a red \( K(1, m_1, ..., m_k) \). Furthermore, since \( v \notin B_i \), \( v \) has more than \( (l - m_i)/(m_i + 1) \) blue adjacencies in \( A_i \). Hence (i) Each vertex in \( B_{k+1} \) has more than \( l - m_k + (l - m_i)/(m_i + 1) = l(m_i + 2)/(m_i + 1) - c_1 \) blue adjacencies in \( A_i \), where \( c_1 \) is independent of \( n \) and \( l \). To this, we add three more observations: (ii) Each vertex in \( E \) has at least \( l - (m_i - 1) \) blue adjacencies in \( A_i \). (In fact, we have seen that this is true of every vertex in \( B_i \).) (iii) Each vertex in \( B_{k+1} \) has at least \( l \) blue adjacencies in \( A_i \). (iv) Each vertex in \( A \) has at most \( (n - 1) - (l - 1) = n - l \) blue adjacencies in \( B_{k+1} \). Therefore

\[
kl(n - l) \geq b_{k+1}(l(m_i + 2)/(m_i + 1) - c_1) + |E|(l - (m_i - 1)) + (M - kl - |E| - b_{k+1})l \\
= b_{k+1}((l/(m_i + 1)) - c) + (M - kl)l - km_il(m_i - 1).
\]
Hence \( b_{k+1} \leq (kn - M + km_l^2 - km_l)(l/(l/(m_1 + 1)) - c_1) \). Replacing \( M \) by its value we obtain, for large \( l \), that there exists a \( c_2 \) such that

\[
b_{k+1} \leq c_2, \text{ where } c_2 \text{ is independent of } n \text{ and } l. \tag{1}\]

Further, since each \( B_i \) contains no red \( K(1, m_1) \), and each vertex in \( B_i \) has at least \( l - (m_1 - 1) \) blue adjacencies in \( A \), it follows that

\[
b_i \leq (n - 1) - (l - (m_1 - 1)) + (m_1 - 1) + 1 = n - l + 2m_1 - 2, \quad 1 \leq i \leq k.
\tag{2}

But we have \( \sum_{i=1}^{k} b_i = M - kl - b_{k+1} \), which differs only by a constant from \( k(n - l) \). Therefore

\[
b_i \geq n - l - c_3, \tag{3}\]

where \( c_3 \) is independent of \( n \) and \( l \).

Suppose that \( b_{k+1} \neq 0 \); that is, suppose there exists a vertex \( v \in B_{k+1} \). Let \( v \) have \( t_1 \) blue adjacencies in \( B \) and \( t_2 \) red adjacencies. Then, using (i) and the fact that \( K(1, n) \not\subset (G)_R \),

\[
t_1 \leq n - 1 - (l(m_1 + 2)/(m_1 + 1) - c_1) \tag{4}
\]

and \( t_2 = M - kl - b_{k+1} - t_1 \).

Let \( D_i, 1 \leq i \leq k \), denote the set of vertices in \( B_i \) to which \( v \) is adjacent in red. By (2) it follows that \( |D_i| \geq t_2 - (k - 1)(n - l + 2m_1) \) for all \( i \). Thus from (1) and (4) there exists a \( c_4 \), independent of \( n \) and \( l \), such that \( |D_i| \geq l/(m_1 + 1) + c_4 \) for all \( i \).

Each vertex in \( B \) has at least \( l - (m_1 - 1) \) blue adjacencies in \( A \) and by (3), \( b_i \geq n - l - c_3 \). Since \( B_i (1 \leq i \leq k) \) contains no red \( K(1, m_1) \), it follows that each vertex of \( B_i \) is adjacent in blue to at most

\[
(n - 1) - (l - (m_1 - 1)) - (b_i - 1 - (m_1 - 1)) = n - l - b_i + 2(m_1 - 1) \leq c_3 + 2(m_1 - 1) = c_5
\]

vertices of \( B_j, j \neq i \). In particular, each vertex of \( D_i \) is adjacent in red to all but at most \( c_5 \) vertices of \( D_j, j \neq i \). Since \( c_5 \) is independent of \( n \) and \( l \), \( l \) is large with respect to \( m_1 \) and \( k \), and \( |D_i| \geq l/(m_1 + 1) + c_4 \) for all \( i \), the graph \( (\bigcup_{i=1}^{k} D_i \cup \{v\})_R \) contains a \( K(1, m_1, m_2, ..., m_k) \), a contradiction.

Thus \( b_{k+1} = 0 \), so that \( lk + \sum_{i=1}^{k} b_i = M \). But then for some \( i, l + b_i = r(K(1, m_1), K(1, n)) \), so that \( (A_i \cup B_i)_R \) contains a \( K(1, m_1) \). This contradiction completes the proof.
3. STAR VERSUS LARGE SPARSE MULTIPARTITE GRAPH

Now we will prove Theorem 2, in which the star is fixed and the complete multipartite graph is large and sparse. Here there are some exceptional cases that are difficult to treat. However, we will deal with some of them in Theorem 2'.

We need a bit more notation. If $D \subseteq V(G)$, we define the neighborhood $N(D)$ of $D$ in $G$ to be the set of vertices in $V(G) - D$ that are adjacent to some member of $D$. Also, if $G$ has been 2-colored, $N_D(D)$ and $N_B(D)$ will denote the neighborhoods of $D$ in the red and blue subgraphs, respectively.

Although the following lemma is trivial, it will be convenient to have it to refer to in what follows.

**Lemma 5.** Let $D$ be a set of $n$ vertices of a graph $G$, with degrees $d_1$, ..., $d_n$. Then $D$ has

$$\sum_{i=1}^{n} d_i - 2q(D))$$

edges joining it to the rest of $G$, and consequently

$$N(D) \leq \sum_{i=1}^{n} d_i - 2q(D).$$

We now establish the lower bound in Theorem 2.

**Lemma 6.** Let $m$ and $n_1$, ..., $n_k$ be fixed, with $m \geq 4$ and $n_i \geq 2$ for each $i$. Then if $n$ is sufficiently large,

$$r(K(1, m), K(n_1, \ldots, n_k, n)) \geq \begin{cases} n + (m - 2) \sum_{i=1}^{k} n_i + k, & \text{if } m \text{ is even and } n + k \text{ is odd,} \\ n + (m - 2) \sum_{i=1}^{k} n_i + k + 1, & \text{otherwise.} \end{cases}$$

**Proof.** Let $M$ denote the appropriate alternative on the right-hand side. Now use Lemma 2 to form a graph $H$ on $M - 1$ vertices with girth at least $m \sum_{i=1}^{k} n_i + 1$, which is $(m - 1)$-regular unless $m$ and $n + k$ are both even, in which case one vertex has degree $m - 2$ and the rest have degree $m - 1$. Color $G = K_{m-1}$ by setting $(G)_R = H$, $(G)_B = \bar{H}$. Clearly $(G)_R$ contains no $K(1, m)$.

Let $D_1$, ..., $D_k$ be any set of pairwise-disjoint sets of vertices in $(G)_R = H$ such that $|D_i| = n_i$ for each $i$, and assume that $D_1$, ..., $D_k$ are the
parts of a $K(n_1, \ldots, n_k)$ in $(G)_R$. Set $D = \bigcup_{i=1}^{k} D_i$. We will show that no $n$ vertices outside of $D$ are simultaneously adjacent in blue to each vertex of $D$, so that the coloring is good. In fact, we will work with $N_{q}(D)$, showing that this has at least $M - n$ vertices. First we observe, in the other direction, that $|D \cup N_{q}(D)| \leq m \sum_{i=1}^{k} n_i$, so $(D \cup N_{q}(D))_R$ is a forest, since $H$ has girth greater than $m \sum_{i=1}^{k} n_i$. Since $(D)_R$ has at least $k$ components, $q((D)_R) \leq p((D)_R) - k = \sum_{i=1}^{k} n_i - k$.

Assume for the moment that either $m$ is odd or $n + k$ is even, so that $(G)_R$ is $(m - 1)$-regular. Then, by Lemma 5, $(D \cup N_{q}(D))_R$ has at least

$$\left(m - 1\right) \sum_{i=1}^{k} n_i - 2q((D)_R) + q((D)_R) \geq \left(m - 2\right) \sum_{i=1}^{k} n_i + k$$

(5)

red edges. Since $(D \cup N_{q}(D))_R$ is a forest, it has at least $(m - 2) \sum_{i=1}^{k} n_i + k + 1 = M - n$ vertices. Returning to the case in which $m$ and $n + k$ are even, we see that the lower bound in (5) must be decreased by 1. But in this case too we see that $(D \cup N_{q}(D))_R$ has at least $M - n$ vertices. Hence in either case we have a good coloring, which completes the proof.

Now we turn to the upper bound in Theorem 2; this will entail careful study of the neighborhoods of sets of vertices, beginning with the following lemma.

**Lemma 7.** Let $m$ and $n$ be integers satisfying $m \geq 4$ and $n \geq 2$, and let $G$ be a graph satisfying $p(G) \geq mn$ and $\Delta(G) < m$. Then there exists a set $D$ of $n$ vertices whose neighborhood satisfies

$$|N(D)| \leq (m - 3)n + 2.$$  

If, in addition, $\delta(G) < m - 1$, then there is such a $D$ with

$$|N(D)| \leq (m - 3)n + 1.$$  

Furthermore, suppose in either case that such a $D$ exists such that equality holds and no $v \in N(D)$ has a neighbor outside $D \cup N(D)$. Then there exists a $D'$ with $n$ vertices for which the corresponding inequality becomes strict.

**Proof.** Choose a component $C_1$ of $G$ which contains a vertex $w$ of degree 5. If $|V(C_1)| \geq n$, let $D$ be any set of $n$ vertices of $C_1$ which includes $w$ and which induces a connected graph. If not, put all of $V(C_1)$ in $D$ and proceed to take vertices from other components $C_2$, $C_3$, ... until $n$ vertices have been taken. The only restrictions are to take all...
the vertices from a component before proceeding to the next, and to take vertices from a connected subgraph of the last component.

Thus, $D = D_1 \cup \cdots \cup D_r$, where $D_i = C_i$, except perhaps for $D_r$, which induces a connected subgraph of $C_r$. By construction, $N(D) = N(D_r)$ and $q(D_r) \geq p - 1$, where $p = |D_r|$. Let the vertices of $D_r$ have degrees $d_1, \ldots, d_p$. Then, by Lemma 5,

$$|N(D)| = |N(D_r)| \leq \sum_{i=1}^{k} d_i - 2q(D_r) \leq (m - 1)p - 2(p - 1)$$

$$= (m - 3)p + 2 \leq (m - 3)n + 2,$$

establishing the first part of the lemma. Also, this inequality is strict unless $t = 1$ and $\delta = m - 1$.

Finally, suppose that $|N(D)| = (m - 3)n + 2$, or that $\delta < m - 1$ and $|N(D)| = (m - 3)n + 1$, and that no $v \in N(D)$ has a neighbor outside $D \cup N(D)$. Let $u$ be a vertex in $D \cup N(D)$ of smallest degree, say $d$. Since $d \leq m - 1$, and $D \cup N(D)$ contains $(m - 2)n + 2$ or $(m - 2)n + 1$ vertices, we see that $D \cup N(D)$ contains at least $|D \cup N(D)| - 1 - d \geq n + (m - 3)(n - 1) - 1 \geq n$ vertices distinct from $u$ that are not adjacent to $u$. Let $D'$ be a set of such vertices. Since $D' \cup N(D') \subseteq D \cup N(D)$ and $u \notin D' \cup N(D')$, $|D' \cup N(D')| < |D \cup N(D)|$, proving the last part of the lemma.

The main argument in Theorem 2 is embodied in the following lemma.

**Lemma 8.** Let $m$ and $n_1, \ldots, n_k$ be fixed integers satisfying $m \geq 4$ and $n_i \geq 2$, $1 \leq i \leq k$. Let $M$ satisfy $M > m(\sum_{i=1}^{k} n_i)$. Two-color $G = K_M$ so that $(G)_R$ contains no $K(1, m)$. Then there exists a $K(n_1, n_2, \ldots, n_k)$ in $(G)_R$ with parts $D_1, D_2, \ldots, D_k$, $|D_i| = n_i$, such that $|N_R(\cup_{i=1}^{k} D_i)| \leq (m - 3)(\sum_{i=1}^{k} n_i) + k + 1$. Furthermore, the inequality is strict if $(G)_R$ contains a vertex of degree less than $m - 1$.

**Proof.** The proof will be by induction on $k$. The case $k = 1$ is covered by Lemma 7, so let $k \geq 2$ and assume the result of the lemma holds for all smaller values of $k$. Choose $n_k$ elements $D_k$ so that $N_R(D_k)$ is of minimal cardinality. By Lemma 7 again, with $n = n_k$, $|N_R(D_k)| \leq (m - 3)n_k + 2$, and equality can occur only when each vertex in $(G)_R$ has degree $m - 1$. Set $H = (V(G) - [D_k \cup V_R(D_k)])$.

We first consider the case in which $|N_R(D_k)| = (m - 3)n_k + 2$, or in which $|N_R(D_k)| = (m - 3)n_k + 1$ and $(G)_R$ contains a vertex of degree $< m - 1$. In this case by the last part of Lemma 7 we may assume that some red edge joins $N_R(D_k)$ to $V(H)$. Therefore, $(H)_R$ has a vertex of degree $< m - 1$, and of course, all of its vertices have degree $\leq m - 1$. Apply the induction hypothesis to $H$; this graph has at least
We conclude that $H$ contains a blue $K(n_1, \ldots, n_{k-1})$ with parts $D_1, \ldots, D_{k-1}$ such that $|N_R(\bigcup_{i=1}^{k-1} D_i)| \leq (m - 3) \sum_{i=1}^{k-1} n_i + k - 1$. But then $D_1, \ldots, D_k$ span the blue $K(n_1, \ldots, n_k)$ we seek, and

$$|N_R(\bigcup_{i=1}^{k} D_i)| \leq (m - 3) \sum_{i=1}^{k-1} n_i + k - 1 + (m - 3) n_k + 2$$

$$= (m - 3) \sum_{i=1}^{k} n_i + k + 1,$$

with the upper bound being one less if $(G)_R$ has a vertex of degree $< m - 1$. This completes this case.

Now we consider the case in which $|N_R(D)| \leq (m - 3) n_k + 1$, or one less than this if $(G)_R$ is not $(m - 1)$-regular. In this case, we argue essentially as before. The upper bounds for $|N_R(\bigcup_{i=1}^{k-1} D_i)|$ become greater by one, since now no red edge need join $N_R(D_k)$ to $V(H)$. But the upper bounds for $|N_R(D_k)|$ are now smaller by one, so that the upper bounds for $|N_R(\bigcup_{i=1}^{k-1} D_i)|$ stay the same. This completes the proof.

We are now ready for the proof of Theorem 2.

**Proof of Theorem 2.** Let $M$ denote the appropriate alternative on the right-hand side of the statement. Lemma 6 shows that $M$ is a lower bound for the desired Ramsey number, so it remains to show that $K_M \rightarrow (K(1, m), K(n_1, \ldots, n_{k-1}, n))$ when $n$ is large. So, two-color $G = K_M$ so that it contains no red $K(1, m)$. We now invoke Lemma 8. Thus there exists a $K(n_1, n_2, \ldots, n_k)$ in $(G)_R$ with parts $D_1, D_2, \ldots, D_k$, such that $|N_R(\bigcup_{i=1}^{k} D_i)| \leq (m - 3) \sum_{i=1}^{k-1} n_i + k + 1$. Also $|N_R(\bigcup_{i=1}^{k} D_i)| \leq (m - 3) \sum_{i=1}^{k} n_i + k$ when $M = n + (m - 2) \sum_{i=1}^{k} n_i + k$, since then $(G)_R$ contains a vertex of degree $< m - 1$. Since $M - |N_R(\bigcup_{i=1}^{k} D_i)| > \sum_{i=1}^{k} n_i \equiv n$, $(G)_b$ contains a $K(n_1, n_2, \ldots, n_k, n)$ when $n$ is large. This completes the proof.

As has been said, Theorem 2 does not cover certain cases. We will now see that at least some of these cases can be dealt with. The following theorem removes the restriction that the $n_i$ are at least 2. Note that if $x$ is real, then $\lfloor x \rfloor$ denotes the least integer $\geq x$.

**Theorem 2'.** Let $m, n_1, n_2, \ldots, n_k$ be fixed, with $m \geq 4$. Then if $n$ is sufficiently large,
\( r(K(1, m), K(n_1, n_2, \ldots, n_k, n)) \) 

\[ = \max \left\{ n + (m - 2) \sum_{i=1}^{k} n_i + k + e, \quad m \left[ \frac{n - 1}{m} \right] + m \sum_{i=1}^{k} \left\lfloor \frac{n_i}{m} \right\rfloor + 1 \right\}, \]

where

\[ e = \begin{cases} 
0 & \text{if } m \text{ is even and } n + k \text{ is odd, and either } k = 1 \text{ or } n_i \geq 2 \text{ for some } i, \\
1 & \text{otherwise.} 
\end{cases} \]

The proof of this is tedious and not very illuminating, so it will be omitted. It is not hard to confirm that Theorem 2' reduces to Theorem 2 when all the \( n_i \) are at least 2. It is curious that in Theorem 1 we need a part of size 1, but that in Theorem 2 this causes difficulties.

The remaining case is \( m = 3 \). In fact, \( m = 3 \) causes the only difficulty. If \( m = 1 \) the result is trivial, while if \( m = 2 \), either a result of [3] or a simple direct argument leads to the following: Let \( k^* \) be the number of odd numbers among \( \{n_1, \ldots, n_k, n\} \). Then

\[ r(K(1, 2), K(n_1, n_2, \ldots, n_k, n)) = n + \sum_{i=1}^{k} n_i + \max\{0, k^* - 1\}, \]

which holds even for small \( n \). This is easy to see, since in any good coloring the red subgraph consists of disjoint edges, so that it is essentially unique.

If \( m = 3 \), then in any good coloring the red subgraph consists of disjoint cycles and paths at most. While such a coloring is not unique, it is of such a restricted nature that one would expect the case to be easy to treat. Perhaps it is, but the best that can be said at the moment is that when \( n \) is large,

\[ n + \sum_{i=1}^{k} n_i + k + 1 \leq r(K(1, 3), K(n_1, n_2, \ldots, n_k, n)) \]

\[ \leq n + \sum_{i=1}^{k} n_i + 2k. \]

4. TRIPARTITE GRAPH VERSUS MULTIPARTITE GRAPH

We now will prove Theorem 3. The lower bound will follow from the following well-known result.
Lemma 9 [3]. Let $G$ and $H$ be graphs, where $H$ is connected. Then if $G$ has chromatic number $\chi$ and $H$ has $n$ vertices,

$$r(G, H) \geq (\chi - 1)(n - 1) + 1.$$ 

Our next lemma is a partial extension of Lemma 4. Because it is of some general interest, we state it a little more carefully than needed for the proof of Theorem 3.

Lemma 10. If $l$ and $n_1, n_2, \ldots, n_k$ are fixed, then there is a real $\alpha$, $0 < \alpha < 1$, such that for $n$ sufficiently large

$$r(K(l, l), K(n_1, n_2, \ldots, n_k, n)) = n + O(n^\alpha).$$

Proof. We will show that we may take $\alpha$ to be the same as in Lemma 4 with $k = 2$, although with a large implied constant. The fact that the right-hand side is a lower bound for the Ramsey number is trivial, so we need be concerned only with the upper bound. By Lemma 4, there is a $\beta$ which depends only on $l$ such that for every $N$, $K_N \rightarrow (K(l, l), K(1, N - \beta N^\alpha))$. From this we see that there is a $\gamma$ which is independent of $n$ such that if $n$ is large and $n \leq N \leq 2n$ (say), then

$$K_N \rightarrow (K(l, l), K(1, N - \gamma n^\alpha)). \tag{6}$$

Set $n^* = \sum_{i=1}^k n_i$ and set $N = n + n^* \gamma n^\alpha$. Since $n^*$ is independent of $n$, $N = n + O(n^\alpha)$, and of course $N \leq 2n$ when $n$ is large. We will show that $K_N \rightarrow (K(l, l), K(n_1, \ldots, n_k, n))$. Consider a 2-coloring of $K_N$ and assume that no red $K(l, l)$ occurs. Then by (6) there is a vertex $v_1$ which is adjacent in blue to $N - \gamma n^\alpha$ vertices. Now apply (6) in turn to these $N - \gamma n^\alpha$ vertices; there is a $v_2$ which is adjacent in blue to $N - 2 \gamma n^\alpha$ of these, and of course $v_1$ is adjacent in blue to $v_2$. Continuing in this manner we find vertices $v_1, \ldots, v_k$, mutually adjacent in blue and also each adjacent in blue to the same set of $N - n^* \gamma n^\alpha$ vertices. This yields a blue $K(n_1, \ldots, n_k, n)$, with the first $k$ parts being chosen from the $v_i$. Thus the proof is complete.

We are now ready to prove Theorem 3.

Proof of Theorem 3. Let $M$ denote the right-hand side of the equality. That $r(K(1, 1, m), K(n_1, \ldots, n_k, n)) \geq M$ follows from Lemma 9, so it remains to show that $K_M \rightarrow (K(1, 1, m), K(n_1, \ldots, n_k, n))$. Suppose to the contrary that $K_M \not\rightarrow (K(1, 1, m), K(n_1, n_2, \ldots, n_k, n))$ for $n$ large, and let $G$ denote a good coloring of $K_M$. By Lemma 10 $(G)_R$ contains a $K(l, l)$, where $l$ is large with respect to $m$ and $\sum_{i=1}^k n_i$, but small with respect to $M$. 


Let \( A_1 \) and \( A_2 \) represent the parts of the red \( K(l, l) \). Since \((G)_n\) does not contain \( K(1, 1, m) \) each graph \( A_i \) can contain no red edges. Define \( B_i, i = 1, 2, \) to be the set of all vertices of \( V(G) - (A_1 \cup A_2) \) which are adjacent in blue to all vertices of \( A_i \), except that if a vertex is eligible for both \( B_1 \) and \( B_2 \) we arbitrarily put it in \( B_1 \). Finally set \( B^* = V(G) - (A_1 \cup A_2 \cup B_1 \cup B_2) \).

Observe that each vertex in \( B^* \) is adjacent in red to some vertices of both \( A_1 \) and \( A_2 \), but is adjacent in red to at most \( m - 1 \) vertices in each. Hence, fewer than \( |B^*| m \) red edges join \( A_i \) and \( B^* \). Set \( n^* = \sum_{i=1}^n n_i \) and define \( |B_i| = \max(|B_1|, |B_2|) \). Let \( D \) be an \( n^* \)-element subset of \( A_j \) with the minimum number of red adjacencies to \( B^* \). The number of red edges joining \( D \) and \( B^* \) is less than \( |B^*| m (n^*/l) < |B^*| \), since \( l \) is large relative to \( m \) and \( n^* \). Therefore, all vertices of \( D \) are commonly adjacent in blue to at least \( \|B^*| \) vertices of \( B^* \), as well as to \( B_j \) and to \( A_j \). But \( |A_j \cup B_j| + \|B^*| \geq \|M| \), so we have a blue \( K(n_1, \ldots, n_k, n) \), a contradiction. This completes the proof.

**5. SOME REMARKS**

We will now consider briefly the question of for which other pairs of complete multipartite graphs can the Ramsey number be evaluated exactly. Of course, one might hope to extend some of these results to small \( n \). The general problem appears formidable, even for the cases covered by Theorems 1–3.

In view of Theorem 1, one might ask for an exact evaluation of \( r(K(m_1, \ldots, m_l), K(1, n)) \) for large \( n \) when all the \( m_i \) are at least 2. Unfortunately, this appears to be very difficult, even when the first graph is \( K(2, 2) = C_4 \), and probably more so in general. In fact, since (for some \( c > 0 \)) \( n + cn^{1/3} \leq r(K(2, 2), K(1, n)) \leq n + cn^{1/2} \), (see [1, 8]), it seems likely that the exact value depends on the existence and properties of certain block designs, making a complete evaluation unlikely, even for large \( n \).

Theorem 3 opens up new territory. The most attractive generalizations of \( K(1, 1, m) \) seem to be, in order of difficulty, \( K(1, \ldots, 1, m) \), \( K(1, 1, m_1, \ldots, m_l) \), and \( K(1, m_1, \ldots, m_k) \). It is not at all clear at this point that the last of these, especially, will turn out to be tractable.

Theorem 3, and part of Theorem 1, are of interest not just because they are exact evaluations, but also because of the particular values of the Ramsey numbers in question. To discuss this point we need some background. Let \( G \) be a graph with chromatic number \( \chi \). Let \( s \), the chromatic surplus, be the minimum number of vertices in any color class of any \( \chi \) vertex-coloring of \( G \). A generalization of Lemma 9 (see [1]) states that if \( G \) is such a graph, and \( H \) is a connected graph on \( n \) vertices, where \( n \geq s \), then \( r(G, H) \geq (\chi - 1)(n - 1) + s \). If, in fact, equality
holds, we say that $H$ is $G$-good. In these terms Theorem 3 says that $K(n_1, ..., n_k, n)$ is $K(1, 1, m)$-good when $n$ is large. A consequence of Theorem 1 is that if $m_1 = 1$, then $K(1, n)$ is $K(1, m_1, m_2, ..., m_k)$-good when $n$ is large, and not in general when $m_1 > 1$.

In [1] it is conjectured that if $G$ is any fixed graph and if $H$ is a sufficiently large connected graph of bounded degree, then $H$ is $G$-good. The simplest and sparsest graphs without bounded degree are the stars, so it becomes natural to ask for what $G$ are all large stars $G$-good. Theorem 1 completely settles this question when $G$ is a complete multipartite graph, since, as was shown in [1], $G$ must have only one vertex in its smallest part.

Theorem 3 extends the class $H$ beyond simply stars in this case. These $H$, and of course stars, are subject to a weaker restriction than that of having bounded degree, namely that of being of bounded edge-density. The edge-density of a graph $G$ is defined to be $\max_{G \subseteq G} [q(G)/p(G')]$. In view of Theorems 1 and 3 we give the following conjecture.

**Conjecture.** Let $G = K(1, 1, m_1, ..., m_k)$, and let $H$ be a set of graphs with bounded edge-density; then all sufficiently large members of $H$ are $G$-good.

Another possible direction to extend Theorems 1–3 is to consider $r(G, H)$ when both $G$ and $H$ are large, sparse, complete multipartite graphs.

6. DEDICATION

The authors dedicate this paper to Paul Erdös on his seventieth birthday in appreciation of his many kindnesses.

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