A Good Idea in Ramsey Theory

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Abstract. Burr had the idea of defining the connected graph $H$ to be $G$-good in case the Ramsey number $r(G, H)$ is given by $(\chi(G) - 1)(|V(H)| - 1) + s(G)$, where $\chi(G)$ and $s(G)$ denote the chromatic number and chromatic surplus, respectively, of $G$. This paper surveys what is known at present concerning the occurrence of Ramsey "goodness" and ends with several open questions.

1. Red vs. Blue. The Ramsey number $r(G, H)$ is the smallest $p$ such that in every two-coloring of the edges of the complete graph $K_p$ using colors red ($R$) and blue ($B$), either the red subgraph $\langle R \rangle$ contains $G$ or the blue subgraph $\langle B \rangle$ contains $H$. Attempts to forge a general theory in which the an exact formula for $r(G, H)$ covers a broad range of cases led Burr to introduce the concept of Ramsey "goodness." Specifically, Burr defined the connected graph $H$ to be $G$-good in case $r(G, H) = (\chi(G) - 1)(|V(H)| - 1) + s(G)$, where $\chi(G)$ and $s(G)$ denote the chromatic number and chromatic surplus of $G$, respectively. [See §2 for the definition of chromatic surplus.] This paper surveys research related to this concept carried out by several investigators, including Burr, Erdős and the authors. Throughout, $H$ will taken to be a connected graph of order $n$. When necessary, we will assume that $n$ is large.

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2. In the Beginning. Chvátal’s observation concerning the complete graph-tree Ramsey number is often cited for its seminal effect in generalized Ramsey theory.

Theorem 1 (Chvátal) If $T$ is any tree of order $n$, 
\[ r(K_m, T) = (m - 1)(n - 1) + 1. \]

However, the following result of Bondy and Erdős [1] should be cited as well, since the proof of this theorem highlighted techniques which have been used in many subsequent papers.

Theorem 2 (Bondy and Erdős) If $n \geq m^2 - 2$, then 
\[ r(K_m, C_n) = (m - 1)(n - 1) + 1. \]

To introduce the idea of Ramsey “goodness,” we begin by defining the chromatic surplus of a graph.

Definition 1 Let $\chi(G)$ denote the chromatic number of $G$. The chromatic surplus of $G$ is the largest number $s = s(G)$ such that in every vertex coloring of $G$ using $\chi(G)$ colors, every color class has at least $s$ members.

The following result of Burr [2] generalizes an observation of Chvátal and Harary [12].

Theorem 3 (Burr) If $H$ is any connected graph of order $n \geq s(G)$, then 
\[ r(G, H) \geq (\chi(G) - 1)(n - 1) + s(G). \]

The example which establishes this bound is extremely simple.

Example 1 With $p = (\chi(G) - 1)(n - 1) + s(G) - 1$, two-color the edges of $K_p$ so that $\langle B \rangle$ consists of $\chi(G)$ disjoint complete graphs, $\chi(G) - 1$ of order $n - 1$ and one of order $s(G) - 1$. In this two-coloring, there is no connected blue subgraph of order $n$ and there is no red copy of $G$.

To see that there is no red copy of $G$, note that an embedding of $G$ into the red subgraph would constitute a partition of the vertices of $G$ into $\chi(G)$ independent sets with one of the sets containing at most $s(G) - 1$ vertices. In light of Theorem 3, Burr made the following definition [2].
Definition 2 A connected graph $H$ of order $n \geq s(G)$ is called $G$-good if

$$r(G, H) = (\chi(G) - 1)(n - 1) + s(G).$$

For the case of $G = K_m$, we simply say that $H$ is $m$-good.

3. Tools. The following lemma is a key element of many proofs of goodness for sparse graphs. A version of this lemma can be found in the paper of Bondy and Erdős [1].

Lemma 1 (Path Extension) Suppose that the edges of a complete graph are two-colored ($R$ and $B$) and assume that $(x_1, x_2, \ldots, x_a)$ is a path of length $a - 1$ from $x_1$ to $x_a$ in $\langle B \rangle$. Let $Y = \{y_1, y_2, \ldots, y_b\}$ be the set of vertices not on this path. If $a \geq b(c - 1) + r$ then at least one of the following must occur.

1. There is a path of length $a$ from $x_1$ to $x_a$ in $\langle B \rangle$.
2. $K_c \subset \langle R \rangle$
3. There is a $r$-element subset of $X = \{x_1, x_2, \ldots, x_a\}$ which is completely joined in $\langle R \rangle$ to $Y$.

Definition 3 A path $(z_0, x_1, \ldots, x_k)$ in a graph $H$ is said to be suspended if each of its internal vertices $x_1, x_2, \ldots, x_{k-1}$ has degree two in $H$.

Lemma 1 is particularly useful in proving that $H$ is $G$-good in those cases where $H$ has an appropriately long suspended path. [See especially Theorem 4.] Proofs of goodness results for trees are aided by the following trichotomy.

Lemma 2 (The Tree Trichotomy) Every tree of order $n \geq 4rst$ has at least one of the following:

1. a suspended path on $r$ vertices,
2. a set of $s$ independent end-edges,
3. a star consisting of $t$ end-edges.

To prove that $r(G, T) = (\chi(G) - 1)(n - 1) + s(G)$ for a given graph $G$ and all trees of order $n$, one relies on various algorithms for embedding $T$ into $\langle B \rangle$, assuming that $\langle R \rangle$ contains no copy of $G$. Use of the tree trichotomy is one such embedding scheme, but there are others. Useful tree embedding algorithms include
(1) Trichotomy,
(2) Greedy algorithm,
(3) Exchange,
(4) \((1/3, 2/3)\) divisions.

The last item refers to the fact that by deleting the appropriate edge or vertex from a tree, we can control the orders of the resulting components. The following result is used in [15].

Lemma 3 Let \(T\) be a tree of order \(n\). Then one of the following occurs.

(1) There exists an edge \(e\) of \(T\) such that the order of each component of \(T - e\) is \(\leq \lfloor 2n/3 \rfloor\).
(2) There exists a vertex \(v\) of \(T\) such that the order of each component of \(T - v\) is \(\leq \lfloor n/3 \rfloor\).

4. Using the Tools. To illustrate the applicability of Lemma 1 and the other tools in §3, we give some representative proofs. The first theorem was proved by Burr in [2]. Using this result, he proved the striking fact that for every graph \(G\) and connected graph \(H_0\), there is an number \(n_0(G, H_0)\) such that every graph \(H\) of order \(n \geq n_0\) which is homeomorphic from \(H_0\) is \(G\)-good. The proof of this theorem relies heavily on Lemma 1 or the "long suspended path" argument.

Theorem 4 (Burr) Let \(G\) be any graph and suppose that \(H_0\) is a connected graph of order \(k\). Choose an edge of \(H_0\) and form the sequence of graphs \(H_0, H_1, \ldots\) in which \(H_j\) is obtained by adding \(j\) new vertices which subdivide the chosen edge. Then \(H_{n-k}\) is \(G\)-good for all sufficiently large \(n\).

Proof. The proof is by induction on \(\chi(G)\). If \(\chi(G) = 1\) then \(G = K_1\) and the result is trivial. [Although this case violates the usual convention under which neither \(G\) nor \(H\) have isolated vertices, it does provide a valid anchor for the induction.] Now assume \(\chi(G) > 1\) and color the vertices of \(G\) with \(\chi(G)\) colors so that the color classes \(C_1, C_2, \ldots, C_\chi\) satisfy

\[ s = |C_1| \leq |C_2| \leq \cdots \leq |C_\chi|. \]

Let \(G'\) be the graph obtained from \(G\) by deleting \(C_\chi\). Then \(G'\) has chromatic number \(\chi(G) - 1\) and chromatic surplus \(s(G)\). Set

\[ c = |V(G)|, \quad r = |C_\chi|, \quad b = c - r \quad \text{and} \quad a = b(c - 1) + r. \]
Set \( p = (\chi(G) - 1)(n - 1) + s(G) \) and let \((R, B)\) be any two-coloring of the edges of \( K_p \). By taking \( n \) sufficiently large, we ensure that \( p \geq r(G, H_{a-2}) \). Thus, assuming that \((R)\) contains no copy of \( G \) and \((B)\) contains no copy of \( H \), we conclude that for some \( j \) satisfying \( a - 2 \leq j < n - k \), \((B)\) contains \( H_j \) but no \( H_{j+1} \). Select a specific copy of \( H_j \) in \((B)\). Let the subdivided edge (suspended path) in this copy of \( H_j \) be \((u, x_1, x_2, \ldots, x_j, v)\). Since \( |V(H_j)| < n \), there are at least \((\chi(G) - 2)(n - 1) + s(G)\) vertices which are not in this copy. It follows from the induction hypothesis (assuming \( n \) to be sufficiently large) that disjoint from the blue copy of \( H_j \) there is a red copy of \( G' \). Now apply Lemma 1 to the two-colored complete graph spanned by \( \{u, x_1, x_2, \ldots, x_j, v\} \) and the vertices of \( G' \). In view of our choices of \( a, b, c \) and \( r \), alternative (1) yields a copy of \( H_{j+1} \) in \((B)\) and alternatives (2) and (3) each lead to copies of \( G \) in \((R)\). Thus a contradiction has been reached. \( \square \)

The next result [4] uses a variety of methods, including "exchange" and "suspended path" techniques.

**Theorem 5 (Burr, Erdős, Faudree, Rousseau Schelp)** If \( H \) is any connected graph with \( n \geq 4 \) vertices and \( q \leq (17n + 1)/15 \) edges, then \( H \) is 3-good.

The proof of this theorem uses the following lemmas.

**Lemma 4** Let \( H \) be a graph of order \( n \). (a) If \( H' = H - x_0 \), where \( x_0 \) is a vertex of degree \( d \) in \( H \), then

\[
r(K_3, H) \leq \max\{r(K_3, H'), (d + 1)(n - 1) + 1\}.
\]

Consequently, if \( \delta(H) = 1 \) and \( H' \) is 3-good, then \( H \) is also 3-good. (b) Suppose that \((u, v_1, v_2, w)\) is a suspended path of length three in \( H \). Let \( H'' \) be the graph obtained from \( H \) by shortening this suspended path to one of length two. Then

\[
r(K_3, H) \leq \max\{r(K_3, H''), 2n - 1\}.
\]

**Proof.** (a) In a two-colored \( K_p \) where

\[
p = \max\{r(K_3, H'), (d + 1)(n - 1) + 1\},
\]

we may assume a copy of \( H' \) in \((B)\). Select such a copy and note that if there is a vertex exterior to this copy which is adjacent in \((B)\) to the appropriate \( d \) vertices and can therefore play the role of \( x_0 \), then there is a copy of \( H \) in \((B)\). Otherwise, one of the \( d \) vertices has degree at least \( n \) in \((R)\) and there is either a \( K_3 \) in \((R)\) or a \( K_n \) in \((B)\).

(b) In a two-colored \( K_p \) where

\[
p = \max\{r(K_3, H''), 2n - 1\},
\]
we may assume a copy of $H''$ in (B). If the path $u, v, w$ in this copy of $H''$ is extended to a path of length three in (B) by adding an exterior vertex, then (B) contains $H$. Let $X$ denote the set of exterior vertices and consider the red graph induced by $X$. Let $x_1x_2$ be any edge of this graph and suppose that there is no path extension involving either $x_1$ and $x_2$ and there is no $K_3$ in (R). Since there is no $K_3$ in (R), we may assume that $x_1v \in B$. Then the six edges joining $\{x_1, x_2\}$ and $\{u, v, w\}$ are completely determined. In particular, $x_2u \in B$, $x_2v \in R$ and $x_2w \in B$. If the red graph induced by $X$ contained an edge $x_3x_4$ which is independent from $x_1x_2$, then (with $z_4$ playing the role of $x_2$ in the argument just made) we would have $z_4u \in B$, $z_4v \in R$ and $z_4w \in B$. In this case, $x_2z_4 \in R$ yields a $K_3$ in (R) and $x_2x_3 \in B$ extends the $u - w$ path to $u, x_2, x_4, w$ and yields a copy of $H$ in (B). Thus, we may assume that the red graph induced by $X$ is a star. Let $x_0$ be the center of this star. Since $x_0$ is adjacent to either $u$ or $v$ in (R), the fact that there is no $K_3$ in (R) means that one of these vertices ($u$ or $v$) is adjacent in (B) to every other vertex to which $x_0$ is adjacent in (R). This produces a $K_n$ in (B) and so a contradiction. □

Lemma 5 Let $H$ be a connected graph with $j \geq 4$ vertices and $j + k$ edges, where $k \geq 1$. If $H$ has no vertex of degree one and no suspended path of length three, then $j \leq 5k$ and this bound is sharp.

Proof. For each vertex of degree two in $H$, replace the associated path of length two by an edge. If the resulting multigraph has $h$ vertices, then it has $h + k$ edges. Since this multigraph contains no vertices of degree less than three, it follows that $3h \leq 2(h + k)$, so $h \leq 2k$. Restoring the original vertices of degree two (at most one for each edge of the multigraph), we see that $H$ has at most $2k + 3k = 5k$ vertices. The example of $K_{2,3}$ shows that this bound cannot be improved in general. □

Lemma 6 If $H$ is a graph with $n \geq 2$ vertices and $q$ edges, then

$$r(K_3, H) < 2q + n.$$ 

Proof. The proof is by induction on $n$, with the $n = 2$ case being immediate ($r(K_3, K_2) = 3 < 4$). Now with $d = \delta(G) \leq \lceil 2q/n \rceil$, let $H'$ be obtained by deleting a vertex with degree $d$ and apply Lemma 4 to get

$$r(K_3, H) \leq \max\{2(q - d) + (n - 1), (d + 1)(n - 1) + 1\}$$

$$< 2q + n.$$ 

This completes the proof by induction. □

Remark. Harary conjectured that

$$r(K_3, H) \leq 2q + 1$$
for all graphs \( H \) with \( q \) edges, and this was proved recently by Sidorenko [18].

**Proof of Theorem 5.** Let \( H \) be a connected graph with \( n \geq 4 \) vertices and \( n + k \) edges. For the cases \( k = -1 \) and \( k = 0 \), the fact that \( H \) is 3-good follows immediately by induction using Lemma 4. Now suppose that \( 1 \leq k \leq (2n + 1)/15 \) and \( r(K_3, H) > 2n - 1 \). By repeated use of Lemma 4, we obtain a connected graph \( H' \) which satisfies \(|E(H')| - |V(H')| = k\) and \( r(K_3, H') > 2n - 1 \), and which has no vertex of degree one and has no suspended path of length three. Let \( j \) denote the number of vertices of \( H' \). Then \( 4k \leq j \leq 5k \). The upper bound follows from Lemma 5 and the lower bound holds since, otherwise, \( r(K_3, H') < 2(k + j) + j < 14k - 3 < 2n - 1 \) by Lemma 6. Let \( l \) denote the number of vertices of degree two in \( H' \). Since \( H' \) has \( j \) vertices, \( j + k \) edges and no vertices of degree one, we have \( 2l + 3(j - l) \leq 2(j + k) \) and so find \( l \geq j - 2k \geq 2k \). Since \( H' \) has no suspended path of length three, vertices of degree two are non-adjacent. If all vertices of degree two were deleted from \( H' \), the final graph \( H'' \) would have \( j - l \leq 3k \) vertices and \( j + k - 2l \leq 2k \) edges and so satisfy \( r(K_3, H'') < 2(2k) + 3k < 2n - 1 \) by Lemma 6. Thus, there must exist a graph \( H'' \) satisfying \(|V(H'')| \leq 5k \) and \( r(K_3, H'') > 2n - 1 \) which contains no vertices of degrees one and for which the deletion of some vertex \( x_0 \) of degree two yields \( r(K_3, H'' - x_0) \leq 2n - 1 \). Since \( 3(5k - 1) + 1 \leq 2n - 1 \), the situation just described contradicts part (a) of Lemma 4. □

**Remark.** In view of the progress made recently by Sidorenko [18], it would seem likely that the result given in Theorem 5 could be strengthened considerably. As the proof stands, however, a stronger result would require an appropriate upgrading of Lemma 4 in addition to Sidorenko’s improvement of Lemma 6.

5. The Present Landscape. Representative results concerning \( G \)-good graphs are shown in Fig. 1. In this table, B, E, F, R, S and Sh stand for Burr, Erdős, Faudree, Rousseau, Schelp and Sheehan, respectively.

In the column headed \( G \), all notation is standard except for the use of \( M(\chi, m) \) in row five of the table.

**Example 2** Let \( M(\chi, m) \) be the graph of order \( m\chi \) and chromatic number \( \chi \) given by

\[
M(\chi, m) = K_m + \cdots + K_m + mK_2.
\]

Thus the vertex set of \( M(\chi, m) \) is partitioned into \( \chi \) \( m \)-element independent sets. One pair of these sets is joined by a matching and the remaining pairs are completely joined.

The result of Burr and Faudree states that \( G \) is a graph with chromatic number \( \chi \) and surplus chromatic surplus 1 such that all large trees are \( G \)-good if and only if \( G \) is a subgraph of \( M(\chi, m) \) for some appropriately large \( m \).
In the column headed $H$, there are conditions which typically require to be large and sparse. In this column, $T$ denotes an arbitrary tree of order $n$, $W_{1,n-1}$ stands for the wheel with $n-1$ spokes and $B_{n-2}$ denotes the book with $n-2$ pages ($B_j = K_2 + K_j$). Also, $q(H)$ denotes the number of edges of $H$ and $\Delta(H)$ denotes its maximum degree.

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Reference</th>
<th>$G$</th>
<th>$H$</th>
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</table>
| B         | [2]       | arbitrary | hom from $H_0$  
  \hline | | | $n \geq n_0$ |
| BE        | [3]       | $K_3$ | $W_{1,n-1}$  
  \hline | | | $n \geq 5$ |
| BEFRS     | [4]       | $K_m$ | $q(H) \leq (17n+1)/15$  
  \hline | | | $n \geq 4$  
  | | | $q(H) \leq n + Cn^\gamma$  
  \hline | | | $\gamma = 2/(m-1)$, $n \geq n_0$  
  | | | $m \geq 4$ |
| EFRS      | [5]       | $C_{2m+1}$ | $q(H) \leq n(1 + \epsilon_m)$  
  \hline | | | $n \geq n_0$ |
| BF        | [10]      | $s(G) = 1$  
  \hline | | | $T$  
  \hline | | | $G \subseteq M(\chi(G), m)$  
  for some $m$  
  | | | $n \geq n_0$ |
| EFRS      | [14]      | arbitrary | $q(H) \leq n + C_1n^\alpha$  
  \hline | | | $\Delta(H) \leq C_2n^\alpha$  
  \hline | | | $\alpha = 1/(2p(G) - 1)$  
  \hline | | | $n \geq n_0$ |
| EFRS      | [15]      | $K_\ell + \bar{K}_m$ | $T$  
  \hline | | | $n \geq 3m - 3$ |
| FRSh      | [17]      | $C_{2m+1}$ | $B_{n-2}$  
  \hline | | | $m \geq 3$  
  | | | $n \geq 4m - 13$ |

FIG. 1
6. The Road Ahead. Generally speaking, the graphs \( H \) in Fig. 1 are either extremely sparse or belong to special families (e.g. wheel, book). Directions for future research include proving that \( H \) is \( G \)-good under much weaker conditions of sparseness and, otherwise, probing the limits of Ramsey goodness. Specifically, we mention the following problems.

Problem 1 Is it true that for fixed \( G \) and \( M \), every sufficiently large connected graph \( H \) satisfying \( \Delta(H) \leq M \) is \( G \)-good?

Problem 2 Let \( f(n) \) denote the largest integer \( q \) such that every connected graph with \( n \) and \( q \) edges is \( 3 \)-good. Does \( f(n)/n \to \infty \) with \( n \)?

Problem 3 Characterize those graphs \( G \) for which every every sufficiently large tree is \( G \)-good. Burr and Faudree have conjectured that \( G \) is such a graph iff \( G \subset M(\chi(G),m) \) for some \( m \). Their result in [10] proves that this is true in case \( s(G) = 1 \).

Problem 4 Is it true that the star is always the "worst" tree, i.e. for each graph \( G \)

\[
r(G,T) \leq r(G,K_{1,n-1})
\]

for every tree \( T \) or order \( n \geq \alpha_0(G) \).

References


